XMLECTURE

10B STANDING WAVE

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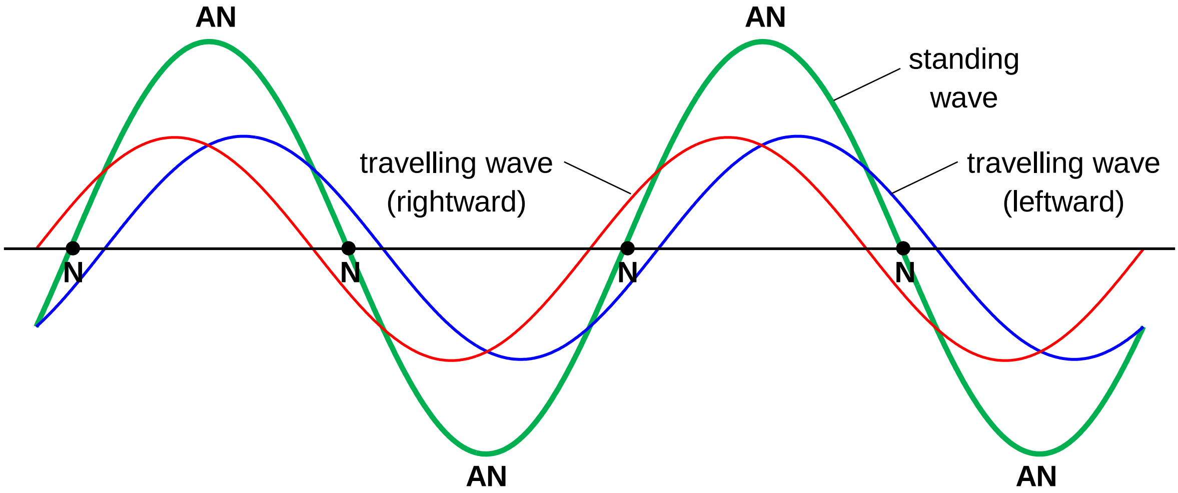
10.7 Standing Wave

Can you imagine what happens when two waves of the same amplitude and wavelength travelling in opposite directions superpose?

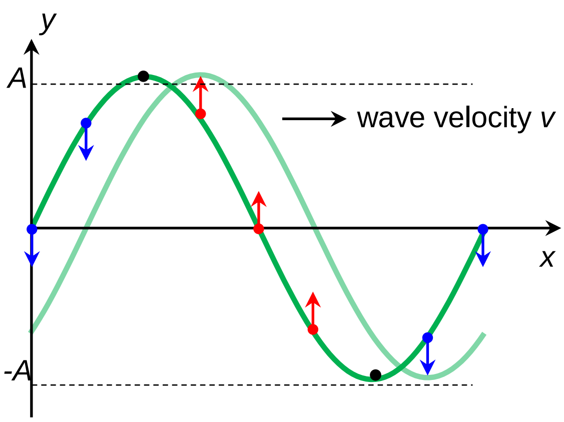


see animation at xmphysics.com

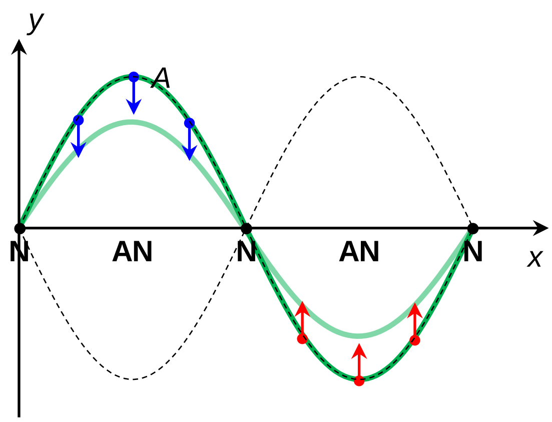
As the two waves move towards each other, the interference alternates between constructive and destructive interference. The resulting wave does retain the frequency and wavelength of the superposing waves. But what happens to the speed?

Propagation

The wave crest of a travelling wave advances. Not so for a standing wave. In fact, overall, there is no net transfer of energy or momentum in either direction. In many situations, the energy is actually trapped between the ends of the standing wave.



travelling wave



standing wave

Amplitude

The amplitude of oscillation is uniform along a travelling wave. Not so for the standing wave. The locations with the maximum amplitude are called the **antinodes** (AN), while the locations with zero amplitude are called the **nodes** (N)[1]. We call the segment between two adjacent nodes, well, a **segment**. A segment corresponds to half a wavelength.

Phase

There is a progressive phase lag along a travelling wave. Not so for the standing wave. All the oscillations in the same segment are completely in phase. This reminds me of the movement of the aunties in a line dance. The aunties may have different amplitudes (because some aunties have shorter legs than others), but they do move in sync (I hope)! On the other hand, adjacent segments are in antiphase. So, two points on a standing wave are either completely in phase, or completely out of phase with each other. The phase difference formula $\Delta\theta = \frac{\Delta x}{\lambda} \times 2\pi$ does not apply to standing waves.

The table below provides a summary.

	Traveling Wave	**Standing Wave**
Propagation	Wave crest advances. Energy is transferred in the direction of wave travel.	Wave crest does not advance. Energy is trapped between the ends of the standing wave.
Amplitude	Uniform. Same amplitude at every point in the wave.	Not uniform. Zero at the nodes, maximum at the antinodes.
Phase	Each point progressively lags the preceding point. $\Delta\theta = \frac{\Delta x}{\lambda} \times 2\pi$	All points of the same segment are in-phase. All points of one segment are in antiphase with all points of the adjacent segments. $\Delta\theta \neq \frac{\Delta x}{\lambda} \times 2\pi$

[1] Antinodes are formed at locations where the two waves meet in phase. Nodes are formed at locations where the two waves meet in antiphase.

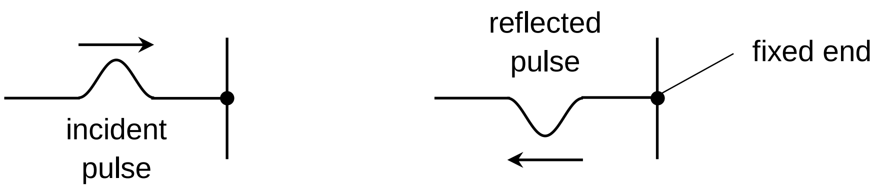
10.8.0 Strings

10.8.1 Reflections

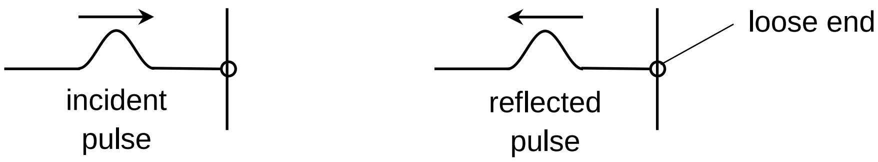
So, a standing wave is formed by superposing two identical waves travelling into each other. You may be wondering, what's the chance of such a thing happening? Actually, it is a common occurrence. One word. Reflection.

see video at xmphysics.com

Imagine that a pulse is sent scrambling towards the end of the slinky. When it reaches the fixed end, it has nowhere to go but return.



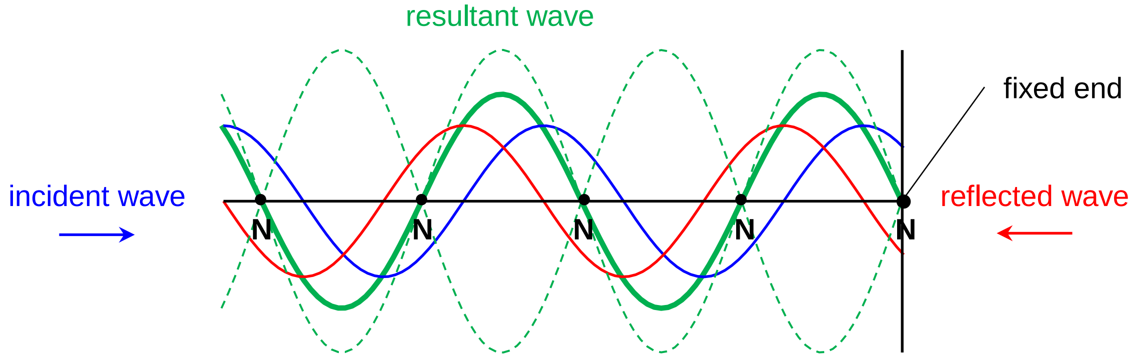
Now, what if instead of a fixed end, we have a loose end? Does the pulse disappear into thin air when it reaches the loose end? Surprise! Whether the end is fixed or loose, the pulse always returns.



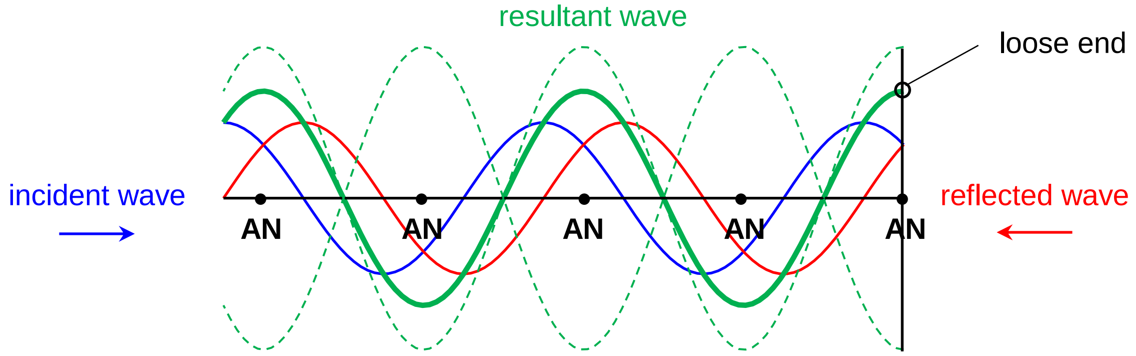
But did you notice there was a difference in the way they returned? With the fixed end, the pulse returns on the opposite side. If you want to sound like an expert, you can say "oh, the pulse underwent a phase change of 180°". With the loose end, the pulse returned on the same side. You can say, "Ooh, this reflection did not incur any phase change."

What if instead of a pulse, we have a continuous sine wave? When the wave arrives at the end (fixed or loose), it undergoes reflection and returns. Voila, the incident wave superposes with the reflected wave to form a perfect standing wave!

If the end is a fixed end, a 180° phase change occurs at the reflection. The incident and reflected waves superpose to form a standing wave with a **node** at the fixed end.

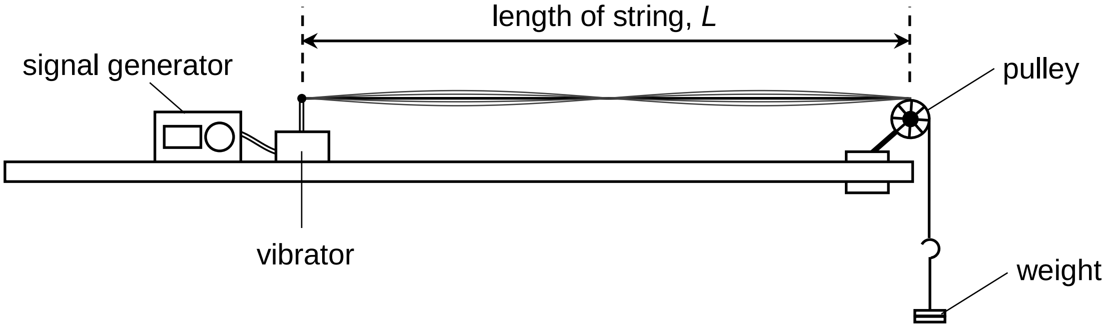


If the end is a loose end, no phase change occurs at the reflection. The incident and reflected waves superpose to form a standing wave with an **antinode** at the loose end.



10.8.2 String Resonance

A vibrator is used to produce waves running along a string. When the vibrator is set to certain frequencies, the string becomes "possessed" and does a vigorous standing wave. Is this black magic or what?



see video at xmphysics.com

Let's try to visualize what's happening in the string. The vibrator produces a sinusoidal wave. This incident wave propagates rightward along the string, undergoes reflection when it reaches the right end, and returns as a reflected wave. When this reflected wave reaches the left end, it undergoes reflection and sets off again as a new incident wave. That's right. The wave is eternally trapped between the two fixed ends of the string as it undergoes reflection repeatedly.

Since the vibrator keeps the wave train continuous, it means that that are numerous incident and reflected waves in the string at the same time. And all these waves superpose, of course. At most frequencies, the interference is destructive, and the resultant wave on the string has negligible amplitude. At some special frequencies, however, constructive interference occurs, and the resultant wave is a standing wave whose amplitude is many times that of the vibrator.[2] [3]

The string is now in **resonance**[4]. And the frequencies at which resonance occurs are called the resonant frequencies.

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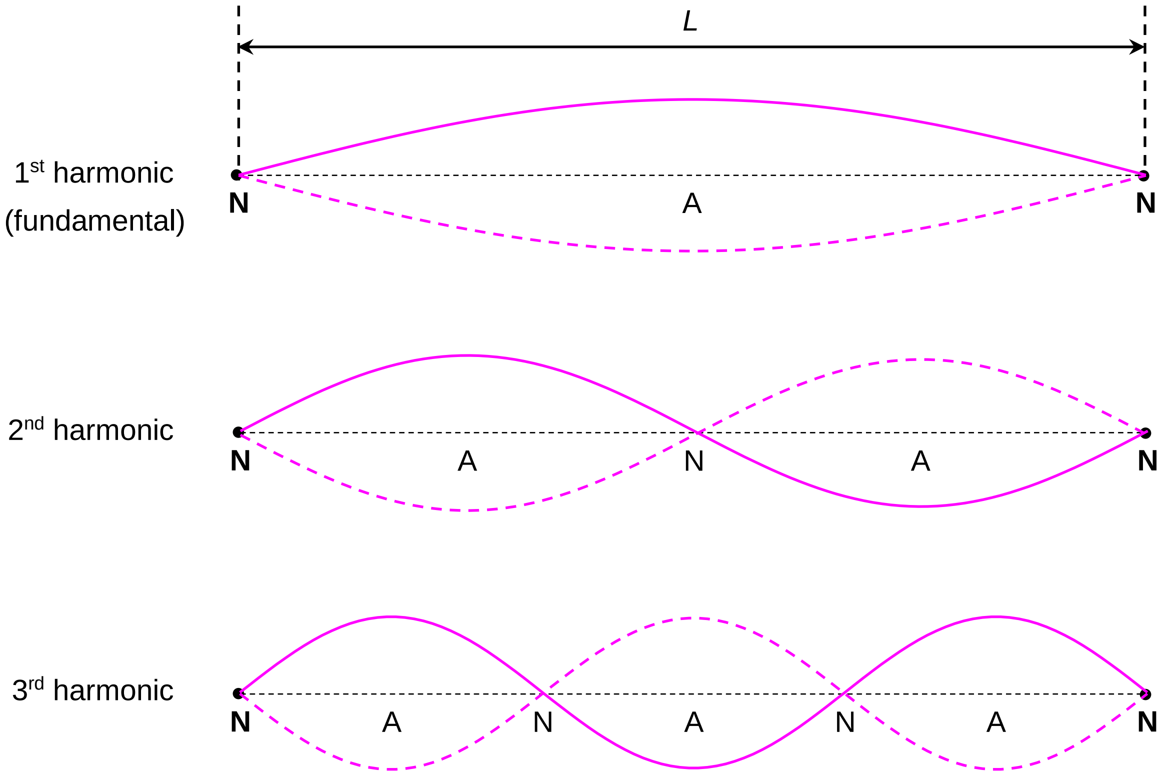
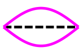
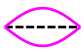
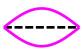
[2] In fact, the amplitude of the standing wave is the sum of the amplitudes of all the incident and reflected waves.
[3] The H2 syllabus does not require you to know the details of the interference among the incident and reflected waves. But it is very interesting and you can read about it in Appendix 10.A.
[4] Because this resonance bears resemblance to the resonance phenomenon in SHM, we call it by the same name. But bear in mind they are not the same thing.

It is actually very easy to work out the values of the resonant frequencies.



As illustrated above, both ends of the string must correspond to nodes on the standing wave. In order to fit two nodes (**N** and **N**) at both ends, we can do **N**A**N**, then **N**ANA**N**, then **N**ANANA**N** and so on.

Basically, we start out with one single  segment, and keep squeezing in another  segment to progress to the next harmonic.

Since each  segment corresponds to half-a-wavelength, we are basically fitting integer number of half-wavelengths onto the string. So

$$n\frac{\lambda_n}{2} = L, \ n = 1, 2, 3, ...$$

This means that the resonant wavelengths are

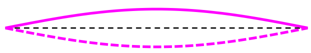
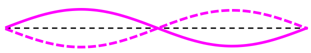
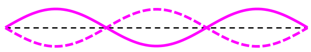
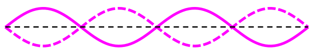
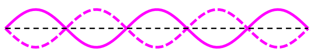
$$\lambda_n = \frac{2L}{n}, \ n = 1, 2, 3, ...$$

Since $v = f\lambda$, the resonant frequencies are

$$f_n = n\frac{v}{2L},\ n = 1, 2, 3, \ldots$$

Let me now introduce some nomenclature: A resonant frequency is also called a harmonic. The lowest resonant frequency, f_1, is also called the fundamental frequency. Since f_n is n times of f_1, f_n is also called the nth harmonic. Harmonics above the fundamental are also called overtones.

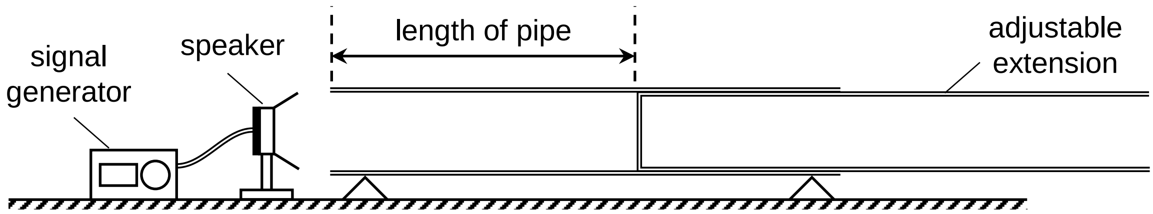
The first 5 harmonics of a string of length L are tabulated below. Notice how the wavelengths of the overtones are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\ldots$ of the fundamental wavelength, and the frequencies of the overtones are 2, 3, 4, 5,… times the fundamental frequency.

Resonance Modes	String Length	Wavelength	Frequency	Nomenclature
	$\left(\frac{\lambda_1}{2}\right) = L$	λ_1	f_1	1st harmonic (fundamental)
	$2\left(\frac{\lambda_2}{2}\right) = L$	$\lambda_2 = \frac{\lambda_1}{2}$	$f_2 = 2f_1$	2nd harmonic (1st overtone)
	$3\left(\frac{\lambda_3}{2}\right) = L$	$\lambda_3 = \frac{\lambda_1}{3}$	$f_3 = 3f_1$	3rd harmonic (2nd overtone)
	$4\left(\frac{\lambda_4}{2}\right) = L$	$\lambda_4 = \frac{\lambda_1}{4}$	$f_4 = 4f_1$	4th harmonic (3rd overtone)
	$5\left(\frac{\lambda_5}{2}\right) = L$	$\lambda_5 = \frac{\lambda_1}{5}$	$f_5 = 5f_1$	5th harmonic (4th overtone)

see video at xmphysics.com

10.9.0 Pipes

In the set up below, the speaker is emitting a monotone. The length of a closed pipe is adjusted by sliding the extension in or out of the pipe. When the pipe is at some particular lengths, the pipe becomes "possessed" and hums loudly in chorus with the speaker. Is this black magic or what?



see video at xmphysics.com

Let's try to visualize what's happening in the pipe. The speaker is emitting a sound wave, a fraction of which is transmitted into the pipe. This sound wave is now kind of trapped in the pipe, undergoing repeated reflections at both ends of the pipe.

Since the speaker is transmitting sound wave into the pipe continuously, it means that the pipe is accumulating incident and reflected sound waves. And all these waves must superpose. At most frequencies, the interference is destructive, and the resultant wave in the pipe has negligible amplitude. At **resonant frequencies**, however, constructive interference occurs. The resultant wave is a standing wave whose amplitude is the sum of the amplitude of all the incident and reflected sound waves in the pipe.[5] The pipe is in **resonance**.

You may be wondering why a sound wave is reflected even by an open end? Good that you're thinking. If you're interested, go read Appendix 10.B. However, the H2 syllabus only requires that you know a standing wave is set up at resonant frequencies, and that

a displacement node/pressure antinode is formed at the closed end, and
a displacement antinode/pressure node is formed at the open end.

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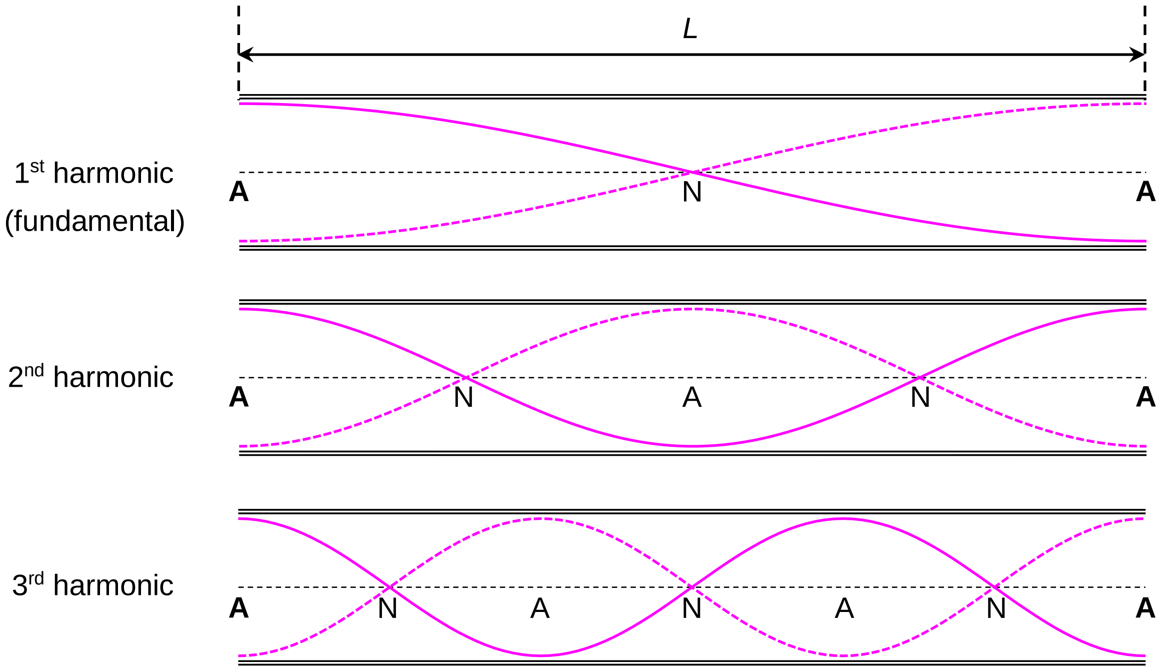
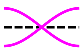
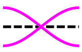
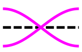
[5] The H2 syllabus does not require you to know the details of the reflection of pressure waves at open and closed ends. But it is very interesting and you can read about it in Appendix 10.B.

10.9.1 Open Pipe

An open pipe is a pipe that is open at both ends. So a displacement antinode must be formed at both ends. To solve for the resonant frequencies, we only have to visualize the wavelengths that will “fit” the pipe and have displacement antinodes at both ends.



In order to fit two antinodes (**A** and **A**) at both ends, we can do **A**N**A**, then **A**NAN**A**, then **A**NANAN**A** and so on. Basically, we start out with one single  segment, and keep squeezing in one more  segment to progress to the next harmonic.

Since each  segment corresponds to half-a-wavelength, we are basically fitting integer number of half-wavelengths into the pipe. So

$$n\frac{\lambda_n}{2} = L, \ n = 1, 2, 3, \ldots$$

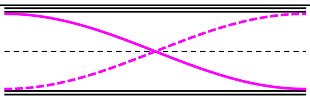
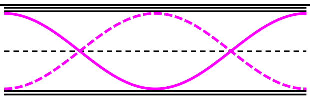
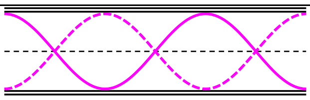
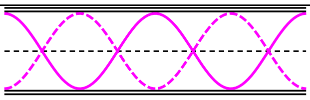
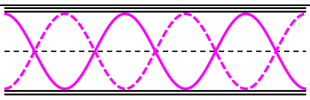
This means that the resonant wavelengths are

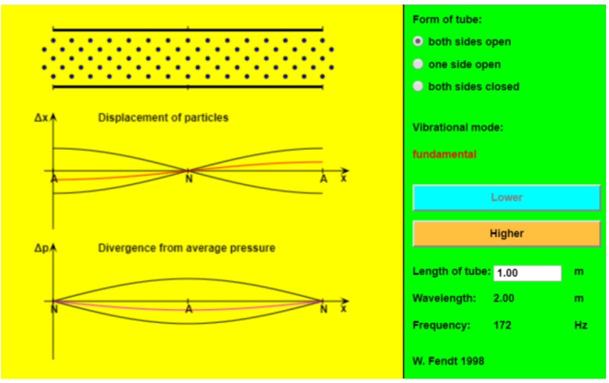
$$\lambda_n = \frac{2L}{n}, \ n = 1, 2, 3, \ldots$$

Since $v = f\lambda$, the resonant frequencies are

$$f_n = n\frac{v}{2L}, \ n = 1, 2, 3, \ldots$$

The first 5 harmonics of an open pipe of length L are tabulated below. Notice how the wavelengths of the overtones are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}...$ of the fundamental wavelength, and the frequencies of the overtones are 2, 3, 4, 5,… times the fundamental frequency.

Resonance Modes	String Length	Wavelength	Frequency	Nomenclature
	$\left(\frac{\lambda_1}{2}\right) = L$	λ_1	f_1	1^{st} harmonic (fundamental)
	$2\left(\frac{\lambda_2}{2}\right) = L$	$\lambda_2 = \frac{\lambda_1}{2}$	$f_2 = 2f_1$	2^{nd} harmonic (1^{st} overtone)
	$3\left(\frac{\lambda_3}{2}\right) = L$	$\lambda_3 = \frac{\lambda_1}{3}$	$f_3 = 3f_1$	3^{rd} harmonic (2^{nd} overtone)
	$4\left(\frac{\lambda_4}{2}\right) = L$	$\lambda_4 = \frac{\lambda_1}{4}$	$f_4 = 4f_1$	4^{th} harmonic (3^{rd} overtone)
	$5\left(\frac{\lambda_5}{2}\right) = L$	$\lambda_5 = \frac{\lambda_1}{5}$	$f_5 = 5f_1$	5^{th} harmonic (4^{th} overtone)

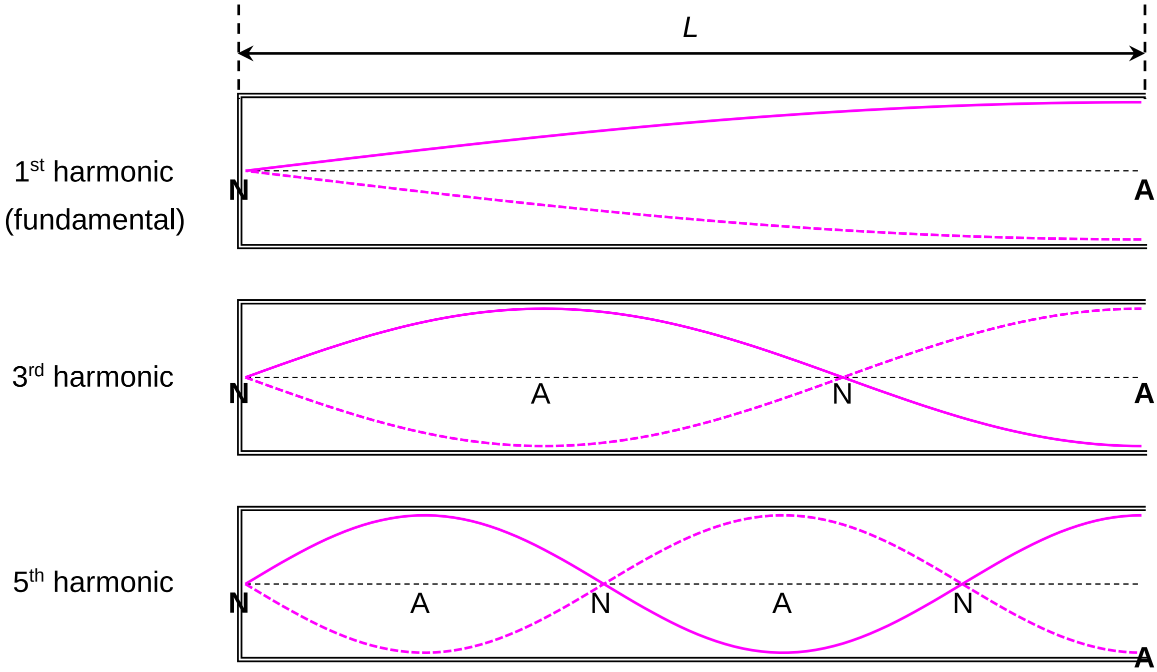


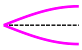
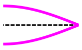
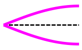
see animation at xmphysics.com

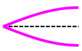
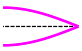
see video at xmphysics.com

10.9.2 Closed Pipe

Firstly, let me clarify that a closed pipe does not refer to a pipe with two closed ends. Instead, a closed pipe refers to a pipe that has one closed end (and one open end). For a closed pipe, a displacement antinode must be formed at the open end and a displacement node must be formed at the closed end.



In order to fit a node and an antinode (**N** and **A**) at either ends, we can do a **NA**, then **N**AN**A**, then **N**ANAN**A** and so on. Basically, we start out with one single  segment, and keep squeezing in one more  and one more  segment to progress to the next harmonic.

Since each  or  segment corresponds to a quarter-wavelength, we are basically fitting odd number of quarter-wavelengths into the pipe. So

$$n\frac{\lambda_n}{4}=L,\ n=1,3,5,7...$$

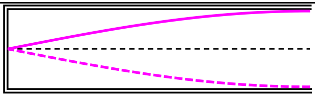
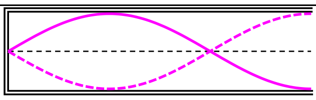
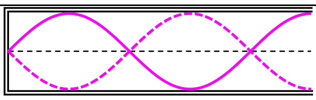
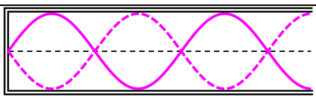
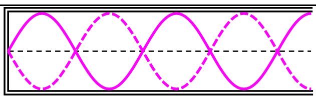
This means that the resonant wavelengths are

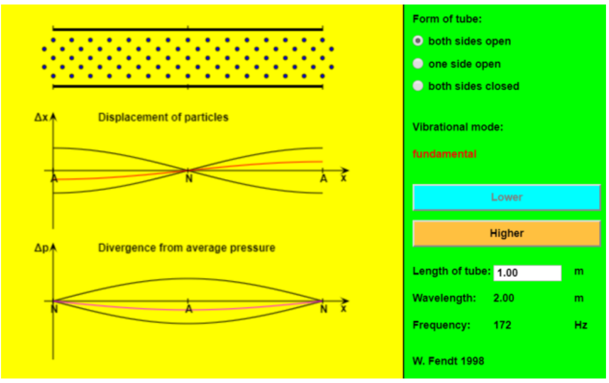
$$\lambda_n=\frac{4L}{n},\ n=1,3,5,7...$$

Since $v=f\lambda$, the resonant frequencies are

$$f_n=n\frac{v}{4L},\ n=1,3,5,7...$$

The first 5 harmonics of a closed pipe of length L are tabulated below. Notice how the wavelengths of the overtones are $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}...$ of the fundamental wavelength, and the frequencies of the overtones are 3, 5, 7, 9,… times the fundamental frequency.

Resonance Modes	String Length	Wavelength	Frequency	Nomenclature
	$\left(\frac{\lambda_1}{4}\right) = L$	λ_1	f_1	1st harmonic (fundamental)
	$3\left(\frac{\lambda_3}{4}\right) = L$	$\lambda_3 = \frac{\lambda_1}{3}$	$f_3 = 3f_1$	3rd harmonic (1st overtone)
	$5\left(\frac{\lambda_5}{4}\right) = L$	$\lambda_5 = \frac{\lambda_1}{5}$	$f_5 = 5f_1$	5th harmonic (2nd overtone)
	$7\left(\frac{\lambda_7}{4}\right) = L$	$\lambda_7 = \frac{\lambda_1}{7}$	$f_7 = 7f_1$	7th harmonic (3rd overtone)
	$9\left(\frac{\lambda_9}{4}\right) = L$	$\lambda_9 = \frac{\lambda_1}{9}$	$f_9 = 9f_1$	9th harmonic (4th overtone)



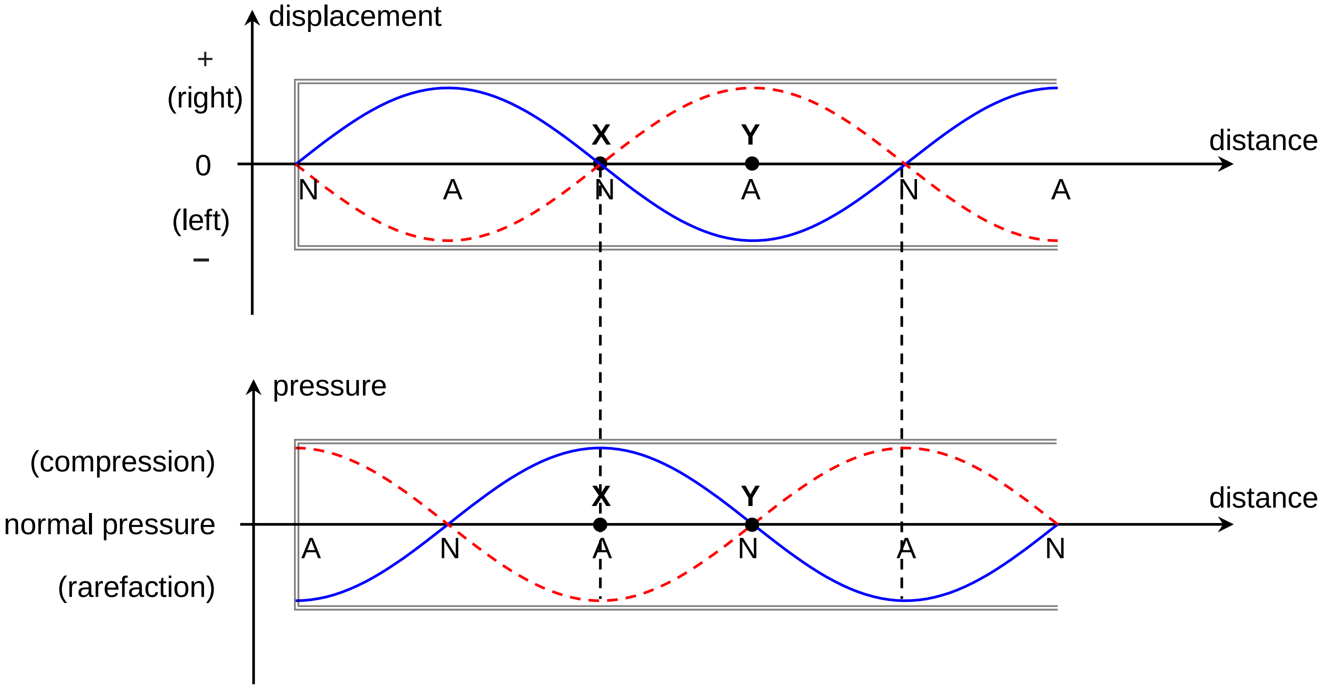
see animation at xmphysics.com

see video at xmphysics.com

10.9.3 Displacement vs Pressure Nodes

A closed end of a pipe is often presented to students as analogous to a fixed end of a string. Since air particles are not able to displace into a closed end, the closed end must be a displacement node (and hence pressure antinode). Likewise, an open end in a pipe is analogous to a loose end of a string. Since an open end allows air particles to displace in and out of the pipe, it must be a displacement antinode (and hence pressure node).

The graphs below shows the displacement and pressure envelope of a standing wave (3rd harmonic) formed in an closed pipe. Graphs are drawn in blue solid lines and red dashed lines to indicate the corresponding pairs in the two graphs. Do you notice that the positions of displacement nodes and antinodes are opposite to the positions of pressure nodes and antinodes?

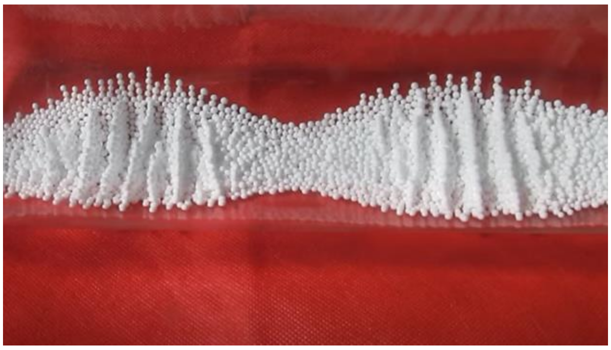


Take the displacement node at position X for example: The displacement profile of the standing wave at this point alternates between  (compression) and  (rarefaction). Basically, the air particles on either sides of this point are always displaced in opposite directions. So they alternate between congregating towards this point and dispersing from this point. This causes a large variation in the pressure here, hence making this point a pressure antinode.

As for the displacement antinode at position Y: The displacement profile at this point is always flat (normal pressure). The air particles around this position always have the same displacement. There is no change to the density of air particles here. The pressure remains constant at normal pressure, hence making this point a pressure node.

10.9.4 More Demonstrations

As you have learnt, standing sound forms pressure nodes and pressure antinodes. Here are a few demonstrations that reveal these nodes and antinodes in rather spectacular fashions.



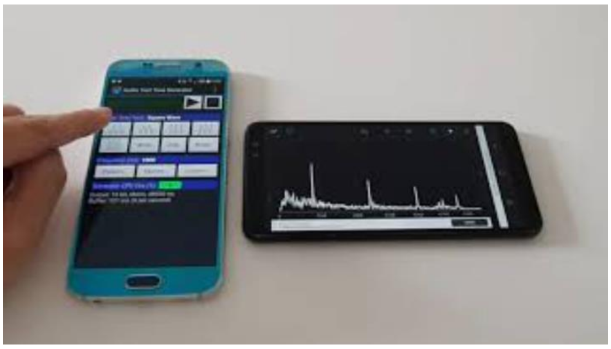
see the Kundt's Tube at xmphysics.com

see the Ruben's Tube at xmphysics.com

10.10 Musical Instruments

A guitar string is a string fixed at both ends. When you pluck a guitar string, it makes a sound of a particular pitch. One single pitch, right?

Nope. Using any free software that analyses the frequency content of a sound, we are presented with evidences that the sound wave produced by a plucked string contains all the harmonics[6]. The lowest frequency harmonic, which usually has the largest amplitude among all the harmonics, is called the fundamental note. Our brains register this as the pitch of the sound. The higher frequency harmonics are called the overtones. The number and relative intensity of the harmonics (called the harmonic content) is one of the primary contributors to the timbre of a sound. The different timbres make sounds of the same pitch produced by different musical instruments sound different to us.



see the video at xmphysics.com

So how is the string able to resonate at so many frequencies at the same time?

When a guitar string is plucked, a triangular pulse is sent running back and forth between the two ends of the string. A triangular pulse is actually made up of sinusoidal waves of a continuous range of frequencies (kind of similar to white light consisting of EM waves of a continuous range of wavelengths). The sinusoidal waves whose frequencies match the resonant frequencies of the string will undergo constructive interference. The sinusoidal waves of the non-resonant frequencies will undergo destructive interference. So the string is acting like a filter. It amplifies its harmonics, but stifles the non harmonics.

Let's move on to the flutes. A flute is basically an open pipe. When you blow across the embouchure hole of a flute, you are energizing the pipe with white noise. As you may guess, white noise contains the entire range of sound frequencies[7]. So you are producing sound waves of all frequencies that travel up and down the pipe as they are reflected at both ends of the pipe. Again the pipe acts like a filter. Sounds waves of the resonant frequencies are amplified through resonance. Sound waves of all the other frequencies undergo mostly destructive interference and are not heard.

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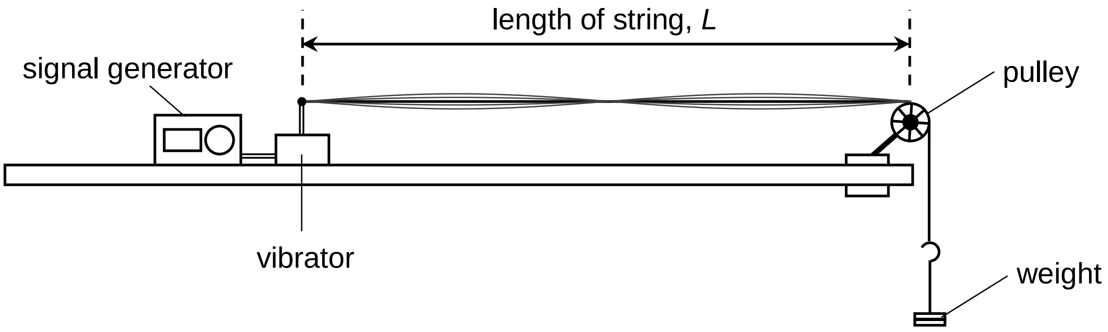
[6] Strictly speaking, the sound wave is produced by the sound board, not by the plucked string. The vibration of the string is picked up by the sound board, whose vibration produces the sound wave that we hear.
[7] It is called white noise because it has all the frequencies, similar to white light having all the wavelengths.

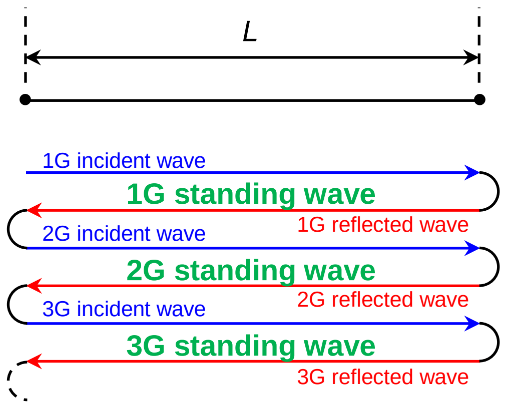
Appendix A: Standing Wave Resonance (in more detail)



What does the H2 syllabus require to know? Not much really. Just that repeated reflections at both ends of the string causes incident and reflected waves to superpose. At resonant frequencies, standing wave of large amplitudes is formed, with nodes at both ends of the string.

I have noticed that even university textbooks do not provide much detail about what's really happening in the string. So I have had to form my own thoughts, which I share in this section.

Let's call the wave that has just departed from the vibrator the 1st generation incident wave. When it returns after the reflection, it becomes the 1st generation reflected wave. The 1st generation incident wave superposes with the 1st generation reflected wave to form, well, the 1st generation standing wave. Assuming no energy loss, the amplitudes of the incident, reflected and standing wave would be *A*, *A* and 2*A* respectively.

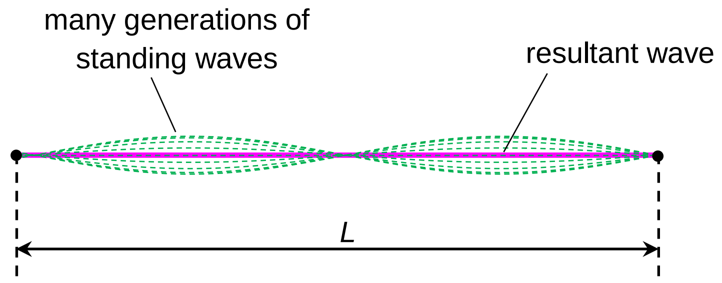


Of course, the 1G reflected wave will undergo reflection and sets off as the 2G incident wave, returns as the 2G reflected wave, superposing to form the 2G standing wave. As long as the vibrator keeps on oscillating the string, the waves in the string will keep piling up. So we have the 1G, 2G, 3G, 4G … *N*G standing waves all in the string at the same time. [8]

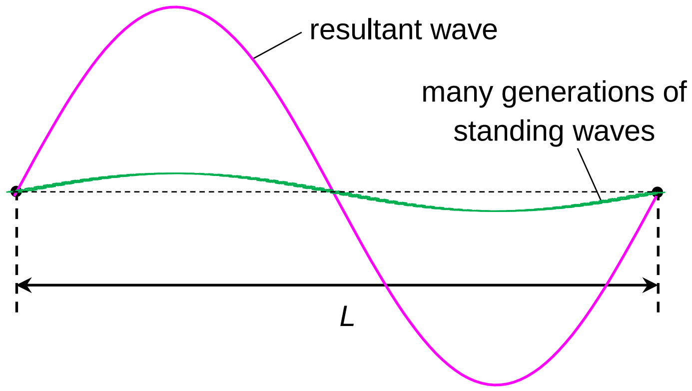
[8] I created this "wave generations" nomenclature myself. You won't read about iterations of waves described as 1G, 2G, 3G and so on in any textbook.

And they all superpose of course.

At most frequencies, all these generations of standing waves would have a progressive phase difference between generations. The resulting destructive interference produces a standing wave of zero (or close to zero) amplitude.



However, at resonant frequencies, all these generations of standing waves would be in-phase with one another. The resulting constructive interference produces a standing wave of amplitude much larger than A. Assuming that there are N generations in the string, and no attenuation occurs during reflections, no energy lost to dissipative force, the amplitude of the standing wave would be $2NA$.



Since every incident wave must go forth and back along the length L of the string before it sets off the next generation wave, the path difference between generations is simply $2L$. For the generations of waves to be in phase,

$$2L = n\lambda,\ n = 1,2,3,...$$

This means that the resonant wavelengths are

$$\lambda = \frac{2L}{n},\ n = 1,2,3,...$$

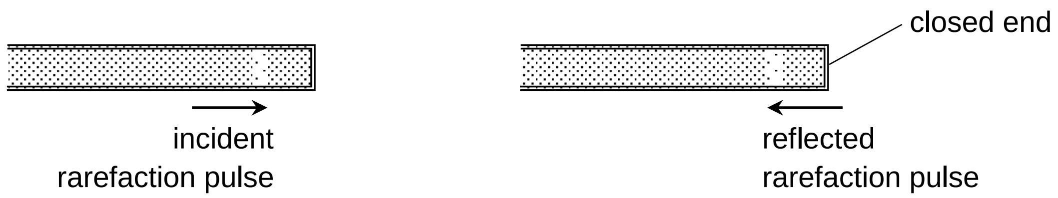
Since $v = f\lambda$, the resonant frequencies are

$$f = n\frac{v}{2L},\ n = 1,2,3,...$$

Appendix B: Pressure Wave Reflections

Let's recall that sound is a longitudinal wave, where the longitudinal oscillation of air particles produces region of compression and rarefaction. Imagine a pressure pulse propagating along a narrow pipe. When it reaches the closed end of the pipe, it must undergo reflection.

The same thing happens if it were a rarefaction pulse.



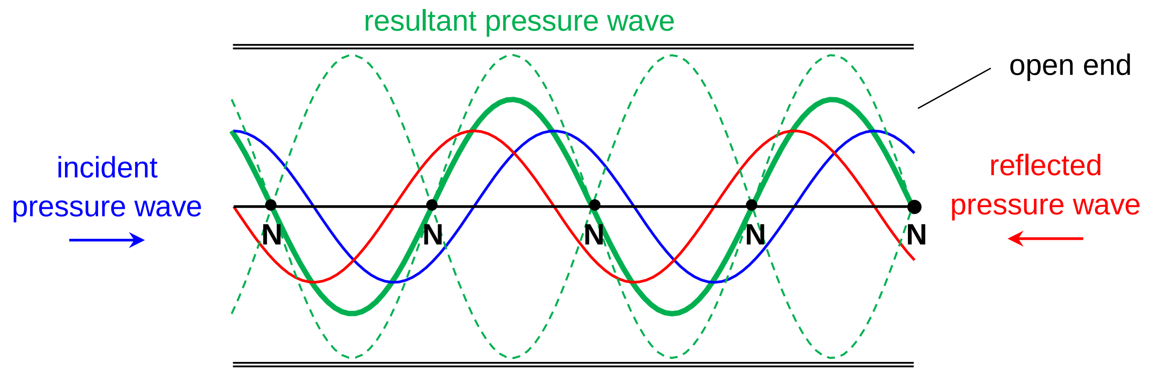
Now, what if instead of a closed end, we have an open end? Does the pulse disappear into thin air when it reaches the open end? Surprise! Whether the end is closed or open, the pulse always returns.

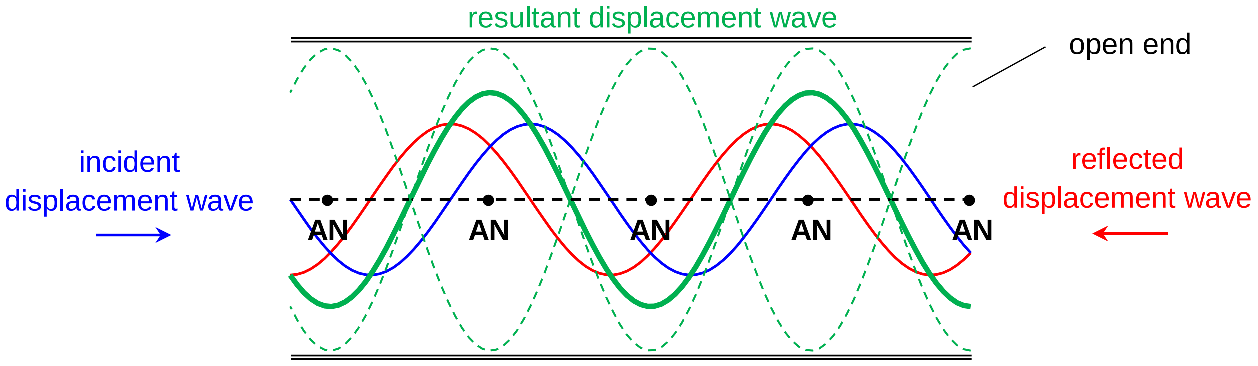
see the animation at xmphysics.com

But there is a major difference in the way the pulses return. With the open end, the compression pulse is reflected as a rarefaction pulse, and vice versa. In other words, the pressure wave undergoes a phase change of 180° at the open end.

What if instead of a pulse, we have a continuous sound wave propagating down the pipe? We have an incident sound wave superposing with a reflected sound wave. Voila, a standing sound wave!

Since a pressure wave undergoes a 180° phase change when reflected at an open end, the incident and reflected pressure wave are always in antiphase at the open end. As such a **pressure node** (and a displacement antinode) is formed at the open end.





Conversely, since a pressure wave is reflected with no phase change at a closed end, the incident and reflected pressure wave are always in-phase at the closed end. As such a **pressure antinode** (and a displacement node) is formed at the closed end.