XMLECTURE

01 MEASUREMENT

NO DEFINITIONS. JUST PHYSICS.

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Online resources are provided at https://xmphysics.com/measurement/

1.1 SI Units

Let me tell you the story of the horse's teeth. A group of scholars are debating sagely over the correct number of teeth in a horse's mouth. A naive boy suggests looking in the horse's mouth. The boy was given a roasting by the scholars for his outrageous suggestion.

A charming parable, isn't it? And it highlights the difference between philosophy and science in their approach to exploring the world. Philosophers base their arguments on reasons and principles. Scientists base their theories upon empirical data.

Galileo is considered by many to be the father of physics. He measured the speed of a ball rolling down a slope to confirm that the speed was increasing at a constant rate. He measured the period of a pendulum to show it is independent of the amplitude of the swing. Measure, measure, measure.

Well, I have digressed too much. Back to the H2 syllabus. To measure quantities, you need standardized units. If the 30 cm on my ruler is longer than the 30 cm on your ruler, how do we know which ruler is good and which is bad? The authority for such matters is the International Bureau of Weight and Measures (BIPM), which administers the International System of Units (SI Units). In total, there are 7 standardized base units for 7 base quantities, as listed in the table below.

Base Quantity	Base Units	
Mass	Kilogram	kg
Length	Metre	m
Time	Second	s
Electric current	Ampere	A
Thermodynamic temperature	Kelvin	K
Amount of substance	Mole	mol
Luminous intensity	Candela	Cd

Did you notice that units such as the Newton and the Coulomb are missing from the list? This is because they are derived units of derived quantities. It can be quite fun to trace a derived unit to its base units. The trick is to think of a formula that links the derived quantity to its base quantities. A few examples are provided below for you to try.

Derived Quantity	Derived Unit	Suggested Formula	Base Units
Force	N	$F = ma$	kg m s^{-2}
Energy	J	$KE = \frac{1}{2}mv^2$	kg m^2 s^{-2}
Charge	C	$I = \frac{Q}{t}$	A s
Voltage	V	$P = VI$	kg m^2 s^{-3} A^{-1}

The standardization of units is actually not a trivial matter. A momentous event happened recently in November 2018, when the world's measurement experts voted to revise the SI: to change from a system that depends on physical objects, to one which is based on physical constants such as the speed of light and the charge of an electron. If you are interested to find out what the fuss is all about, read here. To learn how the SI units are defined now, read here.

1.2 Prefixes

Every day, I dream of winning the one-million-dollar lottery. Then I realize I have to win the lottery every year for 1000 years, before I match the wealth of a nincompoop billionaire like Donald Trump. What?

Anyway, back to our lecture. Physicists poke their noses into everything, from the smallest to the largest. The diameter of a proton is 0.000 000 000 000 0016 m, while Proxima Centauri, the nearest star to Earth, is 40,208,000,000,000,000 m away. To avoid death from counting zeroes, we have invented the standard notation plus the system of prefixes.

Factor	Prefix	Symbol	Order of Magnitude
10^{-15}	femto	f	-15
10^{-12}	pico	p	-12
10^{-9}	nano	n	-9
10^{-6}	micro	u	-6
10^{-3}	milli	m	-3
10^{-2}	centi	c	-2
10^{-1}	deci	d	-1
10^{0}	-	-	0
10^{3}	kilo	k	3
10^{6}	mega	M	6
10^{9}	giga	G	9
10^{12}	tera	T	12

So, let's try again. The diameter of a proton is 1.6 fm, while Proxima Centauri is 4.02×10^{13} km away[1]. Much easier right?

powers of ten

[1] Astronomers don't use Mm, Gm and Tm for astronomical distances. They prefer units like the astronomical unit (symbol AU, 1 AU = 149,597,871 km), based on the Sun-Earth-distance, and or the lightyear (symbol ly, 1 ly = 9.46×10^{12} km), based on distance travelled by light in one year.

1.3 Estimation

As physics students, you are expected to be able to come up with reasonable estimates. The expectation is that you should at least arrive at the correct order of magnitude.

For example, what is the volume occupied by my body? Is it about 10 m^3, 1 m^3, 0.1 m^3 or 0.01 m^3? That's what we mean by orders of magnitude. Some students imagine a cube of side 0.5 m, and say, "Ya, I think Mr. Chua is about $0.5 \times 0.5 \times 0.5 \approx 0.1\,\text{m}^3$." Other students think, "0.1 m^3 is $0.1 \times 10^6 = 100{,}000\ \text{cm}^3$, which is the volume of 100 1-litre coke bottles. Ya, Mr. Chua is about 0.01 m^3 in volume". Yet others may think, "Humans are 60% water, and the density of water is 1000 kg m^{-3}. So 0.1 m^3 of water weighs 100 kg. Mr. Chua looks about that heavy. I will go with 0.1 m^3".

As you can see, estimation problems can be approached from different angles and can be a lot of fun. And having some common sense and some ball park numbers in your head can be very helpful.

The following is a list of useful numbers to have in your head. I am not suggesting that you memorize them. If you start paying attention to the numbers you encounter in your daily life or in your course work, you should be able to develop your own library of ball park numbers.

<u>Length</u>

Radius of Earth	6400 km
MRT train	100 m
Hair thickness	0.1 mm
Size of atom	10^{-10} m
Size of nucleus	10^{-15} m

<u>Speed</u>

Jogging	12 km h^{-1}
Speed limit on roads	50 km h^{-1}
Speed limit on expressways	80 km h^{-1}
Jet plane	900 km h^{-1}
Sound	330 m s^{-1}
Light	3.00×10^8 m s^{-1}

Volume

1 litre bottle	1000 cm^3
Can drink	330 ml

Mass

1 cm^3 of water	1 g
1 m^3 of water	1000 kg
Car	2000 kg
Earth	6.0×10^{24} kg

Density

Water	1 g cm^{-3}, or 1 ton per m^3
Ice	About 90% that of water's
Steel	About 8 times that of water's

Pressure

Atmospheric pressure	1.0×10^5 Pa.
Car tyre	200 kPa

Energy

Specific heat capacity of water	4.2 kJ kg^{-1}.
Specific latent heat of water	330 kJ kg^{-1}.
Specific latent heat of vaporization	2.3 MJ kg^{-1}.

Voltage

Duracell battery	1.5 V
Mains supply	220 V
CRT, X-ray machine	> 1 kV

Power

A domestic light bulb	10 W
Electric Kettle	~ 2 kW
Power plant	~ MW

1.4 Errors and Uncertainties

A student uses an electronic balance to weigh a 1 dollar coin. The reading shown on the balance is 7.6 g, but the actual mass could be anything between 7.55 and 7.65 g. So the resolution of the measuring instrument introduces an **instrumental uncertainty** of 0.05 g. With this in mind, he presents his measurement as $7.60 \pm 0.05\text{ g}$. What about errors then? Strictly speaking, we can talk about errors only if we know what the true or correct value is. For example, through a Google search, we learn that the "correct" mass of Singapore's dollar coin is 7.62 g. So our measurement of 7.60 g corresponds to an **error** of $7.60 - 7.62 = -0.02\text{ g}$.

Another student uses a stopwatch to measure the time taken for a ball to roll down a slope. The reading on the stopwatch shows 3.02 s. The student is aware that as a human there is inconsistency in the delays incurred during the starting and stopping of the stopwatch. As such he timed the motion a few more times, obtaining readings ranging from 2.90 s to 3.10 s, with an averaged value of 3.00 s. He thus estimates that human reaction time introduced a **procedural uncertainty** of 0.1 s to the measurement[2], and presents the measurement as $3.0 \pm 0.1\text{ s}$. If we establish the "true" value to be 2.8 s, through calculating the theoretical time taken to roll down the slope, we can then declare that the measurement of 3.0 s has an **error** of $3.0 - 2.8 = +0.2\text{ s}$.

Finally, regarding the final presentation of the measurement, it is customary to do the following:

1. Round off the uncertainty to 1 significant figure only.
2. Round off the value to the same decimal place as the uncertainty.

1 s.f. ↓

$$4.103 \pm 0.002$$

↶ same d.p.

The examples below should make these two points clear.

9.784 ± 0.028	is to be presented as 9.78 ± 0.03
1.234 ± 0.234	is to be presented as 1.2 ± 0.2
1234 ± 234	is to be presented as $1200 \pm 200 \Rightarrow (1.2 \pm 0.2) \times 10^{2}$
0.12351 ± 0.00098	is presented as $0.124 \pm 0.001 \Rightarrow (1.24 \pm 0.01) \times 10^{-1}$

[2] The instrumental uncertainty (which is 0.005 s) is irrelevant since the procedural uncertainty is much larger.

1.5 Precision and Accuracy

"I would rather be imprecisely accurate, than precisely inaccurate."
~ anonymous lazy experimenter

Roughly speaking, **precision** is related to the amount of uncertainty in a measurement. The smaller the uncertainty, the higher the precision. **Accuracy,** on the other hand, is related to the magnitude of error in a measurement. The smaller the error, the higher the accuracy.

Measurement A: $4.8 \pm 0.1\,\Omega$

Measurement B: $4.57 \pm 0.01\,\Omega$

True value: 4.70 Ω

In the above example, B is more precise than A (since it has an uncertainty of only 0.01 Ω compared to 0.1 Ω for A). But A is more accurate than B (since its error is only 0.1 Ω compared to −0.13 Ω for B)

These two concepts are also applicable when a measurement is repeated to obtain a data set.

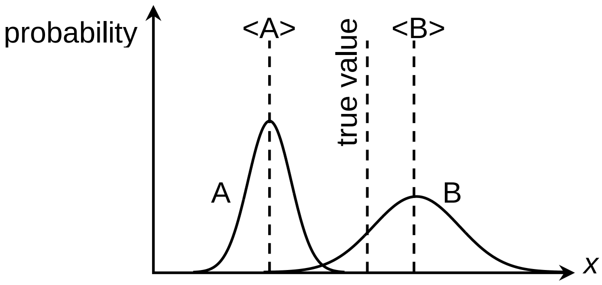
Data set A = {1.000 s, **1.002** s, **0.998** s, 1.001 s, 0.999 s}   Average value, <A> = 1.000 s [3]

Data set B = {**1.032** s, 1.010 s, 1.001 s, 1.018 s, **0.989** s}   Average value, <B> = 1.010 s

True value: 1.008 s

For the example given above, A is more precise since its range ($1.002 - 0.998 = 0.004$ s) is narrower. B however is more accurate since its error ($1.010 - 1.008 = 0.002$ s) is smaller.

If we have a large data set, we can even plot out the distribution curve. With a large population, the graph often shows a normal distribution. The precision will correspond to the width or spread of the graph (standard deviation would be the more technical term), while the accuracy is reflected in the deviation of the mean value from the true value.

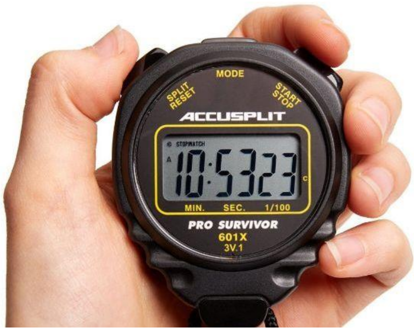


In the example above, A is more precise, but B is more accurate.

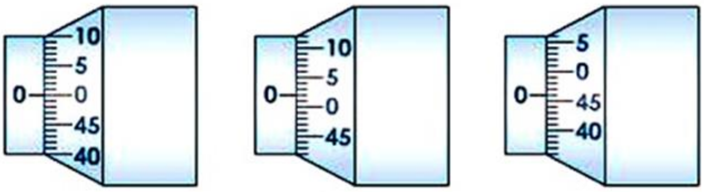
[3] <X> denotes average value of X

1.6 Random and Systematic Errors

Both random errors[4] and systematic errors lower the quality of our measurements. Basically, random uncertainties (whether instrumental or procedural) reduce precision, while systematic errors worsen accuracy. In this section, we will highlight the difference between these two sources of errors.



Random errors are random, unpredictable and therefore not reproducible. Timing the fall of a ball is a very good example of a measurement strewn with random errors. You make repeated measurements and get multiple different readings, and nobody knows for sure which reading is closest to the true value. But if random errors are equally likely to be positive and negative (in many practical situations, random errors are normally distributed), they should add up to zero statistically. For this reason, we are expected to repeat measurements and calculate the average. By repeat-and-averaging, we have a good statistical chance of reducing (we dare not say eliminate) the random errors and improve our chance of getting the true value.



(a) No zero error | (b) Positive zero error | (c) Negative zero error

On the other hand, systematic errors are consistent, predictable and therefore reproducible. The zero error is a classic example. If your micrometer screw gauge has zero error of +0.02 mm, every single reading will be too large by 0.02 mm. Do you realize that systematic errors cannot be reduced by repeat-and-averaging? For this reason, we must be alert to the presence of systematic errors, and take actions to eliminate them once they have been identified.

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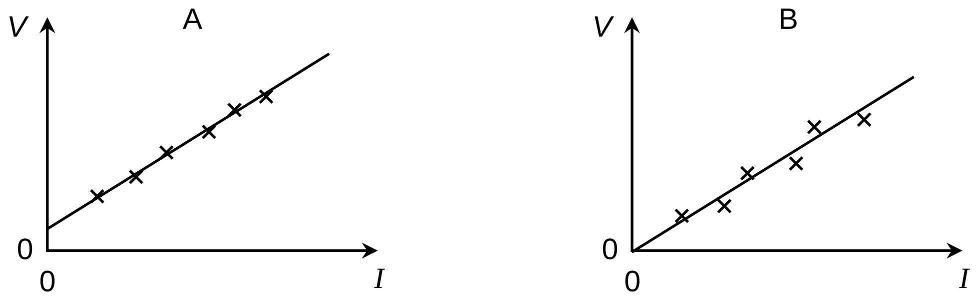
[4] Actually “random uncertainties” is a more correct term. But as a community we have been quite ill disciplined and often use the word “error” when we actually mean “uncertainty”.

Plotting our data on a graph is an effective way to force both random and systematic errors to reveal themselves. For example, when we plot the *V*-*I* graph of a resistor, we expect to obtain a straight line graph passing through the origin (because $V = IR$). We can also calculate the gradient of the graph to obtain the resistance of the resistor.



For the example shown above, the random errors are larger in B than A. This is indicated by the amount of scatter of the data points on both sides of the best-fit-line. But the line does not pass through the origin in A. This suggests the presence of systematic errors in A. (Perhaps the voltmeter has a positive zero error.)

1.7 Propagation of Error/Uncertainty

Let $S = A + B$

If A increases by 2, and B by 3, S would increase by 5. Obvious, right?

Now let $P = A \times B$

If A increases by 2% and B increases by 3%, P would increase by approximately 5%. This one is not as obvious. But if you think $1.02 \times 1.03 = 1.0506$, maybe you will see the light.

$$\uparrow 2\% \quad \uparrow 3\% \quad \uparrow 5\%$$
$$102\% \times 103\% \approx 105\%$$

Anyway, from the insight gained from these simple arithmetic exercises, we can formulate the two rules regarding the propagation of errors or uncertainties.

1. **Summation Rule**: If $S = A + B$, $\quad \Delta S = \Delta A + \Delta B$
   If $D = A - B$, $\quad \Delta D = \Delta A + \Delta B$

   In simple English, this rule is saying that when measurements are summed (or subtracted), the individual **absolute** uncertainties add up to be the resultant **absolute** uncertainty.
   For example, if every month you earn $\$(3000 \pm 500)$ but spend $\$(1000 \pm 400)$, your monthly saving would range from a minimum of $2500 - 1400 = \$1100$ to a maximum of $3500 - 600 = \$2900$. Or you can think $\$(3000 \pm 500) - \$(1000 \mp 400) = \$(2000 \pm 900)$. Either way, the "resultant" uncertainty is $400 + 500 = \$900$. Remember that we always sum up the uncertainties, regardless of whether the measurements were added or subtracted.

2. **Coefficient Scaling Rule**: If $S = kA$, $\quad \Delta S = |k|\,\Delta A$

   This is just an extension of the Summation Rule. It should be quite intuitive, as illustrated in the following example:

   If $S = 2A - 3B + 5$,
   $$S = A + A - B - B - B + 5$$
   $$\Delta S = \Delta A + \Delta A + \Delta B + \Delta B + \Delta B + 0$$
   $$= 2\Delta A + 3\Delta B$$

   Notice that the constant 5 is not a measured value. It does not have any uncertainty.

3. **Product Rule**: If $P = A \times B$, $\frac{\Delta P}{P} = \frac{\Delta A}{A} + \frac{\Delta B}{B}$

If $Q = A \div B$, $\frac{\Delta Q}{Q} = \frac{\Delta A}{A} + \frac{\Delta B}{B}$

In plain English, this rule is saying that when measurements are multiplied (or divided), the **percentage** uncertainties add up to be the resultant **percentage** uncertainty.

For example, if the length L and breadth B of a rectangle are measured to be $L = 2.00 \pm 0.01\,\text{m}$ and $B = 1.00 \pm 0.01\,\text{m}$, the percentage uncertainty in the calculated value of the area would be $\frac{\Delta A}{A} = \frac{\Delta L}{L} + \frac{\Delta B}{B} = \frac{0.01}{2.00} + \frac{0.01}{1.00} = 0.5\% + 1\% = 1.5\%$.

4. **Power Scaling Rule**: If $P = A^k$, $\Rightarrow$ $\frac{\Delta P}{P} = |k| \frac{\Delta A}{A}$

This is just an extension of the Product Rule. The following example should clarify.

If $P = 5A^2 \div B^3$,

$$P = 5 \times A \times A \div B \div B \div B$$

$$\frac{\Delta P}{P} = 0 + \frac{\Delta A}{A} + \frac{\Delta A}{A} + \frac{\Delta B}{B} + \frac{\Delta B}{B} + \frac{\Delta B}{B}$$

$$= 2\frac{\Delta A}{A} + 3\frac{\Delta B}{B}$$

Notice that the coefficient 5 is irrelevant when it comes to percentage uncertainty. Why? Let's say $P = 5A$, so $\frac{\Delta P}{P} = \frac{5\Delta A}{5A} = \frac{\Delta A}{A}$. The coefficient 5 appears in both the numerator and the denominator, and thus cancels itself out as far as $\frac{\Delta P}{P}$ is concerned.

5. **Max-Minus-Min Rule**: $\Delta Z = \frac{Z_{max} - Z_{min}}{2}$

When the calculation involves only summation or only multiplication of measurements, the earlier four rules provide a short cut to obtaining the resultant uncertainty. If the calculation is a mixture of both summation and multiplication of measurements, or involves non-linear functions such as sin, cos, log, etc, we will just calculate the uncertainty the brute-force way. E.g. if $H = L\cos\theta$, and L and θ are measured to be $L = 2.0 \pm 0.2\,\text{cm}$ and $\theta = 30 \pm 2°$, then

$$H_{max} = 2.2\cos 28° = 1.942$$

$$H_{min} = 1.8\cos 32° = 1.526$$

$$\Delta H = \frac{1.942 - 1.526}{2} = 0.208 \approx 0.2\ \text{cm}$$

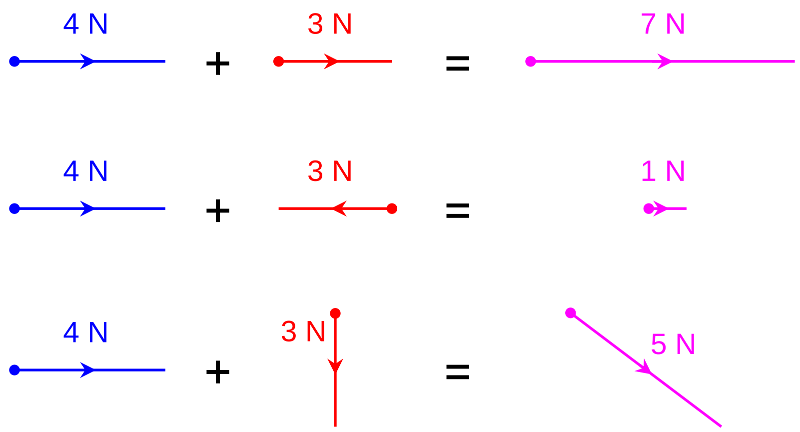
1.8 Vectors

Scalars are quantities with magnitude only: distance, speed, mass, etc.
Vectors are quantities with both magnitude and direction: displacement, velocity, force, etc.

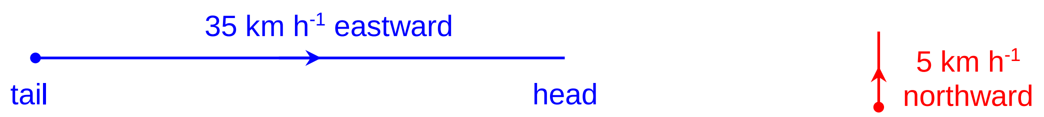
Sometimes a sign convention is adopted to denote the direction of a vector. For example, +1.0 m s^{-1} and -1.0 m s^{-1} may imply velocities of 1.0 m s^{-1} in the rightward and leftward direction respectively. Because of this, some students think vector the moment they see signs. However, we can have temperatures of +10°C and -10°C, and GPE can be +10 J or -10 J. But temperature and energy are scalar quantities. There is no such thing as a rightward or leftward temperature. Neither is there a northward or southward GPE. So, don't be duped by plus and minus signs that do not denote any spatial directions.

1.8.1 Vector Diagram

Scalar addition is straight forward: 3 + 4 is always equal to 7. Vector addition is slightly trickier. Take for example the summation of a 3 N and a 4 N force. If they are in the same direction, the resultant force is 7 N. If the two forces are in opposite directions, the resultant force is 1 N. If they are perpendicular to each other, the resultant is a 5 N. In fact, depending on the direction of the two vectors, 3 N + 4 N can be equal to anything between 1 N and 7 N.



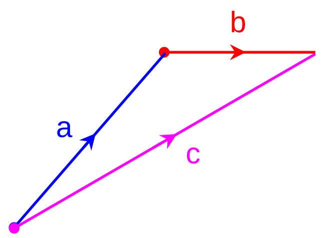
From your secondary school math, you may have learnt vector notations such as $\overrightarrow{AB}$, $\underset{\sim}{a} - \underset{\sim}{b}$ $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$, $3\mathbf{i} + 4\mathbf{j}$, etc. Good news. You won't encounter any of these in A-level Physics. The most complicated thing you'll be made to do, is drawing a few arrows (the resulting diagram is called a vector diagram) and solving a few triangles.



Just to jog your memory. In vector diagrams, each vector is represented by an arrowed line: the length indicates the magnitude, the arrow indicates the direction.

To add two vectors **a** and **b** together, we arrange **a** and **b** head-to-tail, and draw the resultant vector **a+b** from the free tail to the free head. To subtract **b** from **a**, we arrange **a** and **−b** (meaning we have to flip **b** 180° around) head-to-tail, and draw the resultant vector **a−b** from the free tail to the free head.

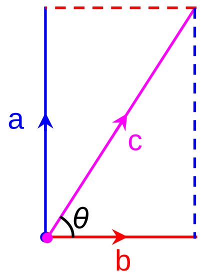
When you're more confident, you can develop other ways of doing vector summation. For example, keeping the tails of the vectors together, you can obtain **a** + **b** and **a** – **b** in the diagonals of the parallelogram.



$$c^2 = a^2 + b^2 - 2ab\cos\hat{c}$$

$$\frac{a}{\sin\hat{a}} = \frac{b}{\sin\hat{b}} = \frac{c}{\sin\hat{c}}$$

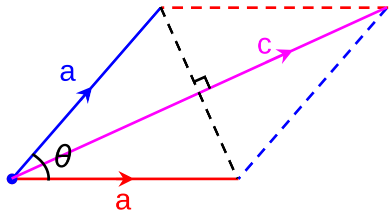
Technically, if you construct your vector diagrams accurately enough with rulers and protractors, you can obtain the results by measuring the length of the resultant vector. But since you have learnt enough mathematics, it is more convenient to do a decent sketch of the vector diagrams but solve them accurately through calculations, typically using cosine rule or sine rule.



$$c^2 = a^2 + b^2$$

$$\tan\theta = \frac{a}{b}$$

Often, in H2 physics we are working with perpendicular vectors. So our parallelograms and triangles become rectangles and right-angled triangles. In these cases, we can dump the sine and cosine rules, and use Pythagoras and trigo ratios instead.



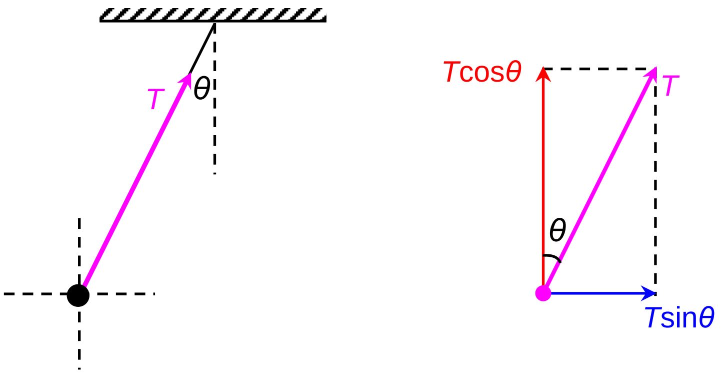
$$c = 2a\cos\frac{\theta}{2}$$

At other times, we are working with vectors of the same magnitude, in which case our parallelograms and triangles become rhombuses and isosceles triangles, which are again relatively easy to solve.

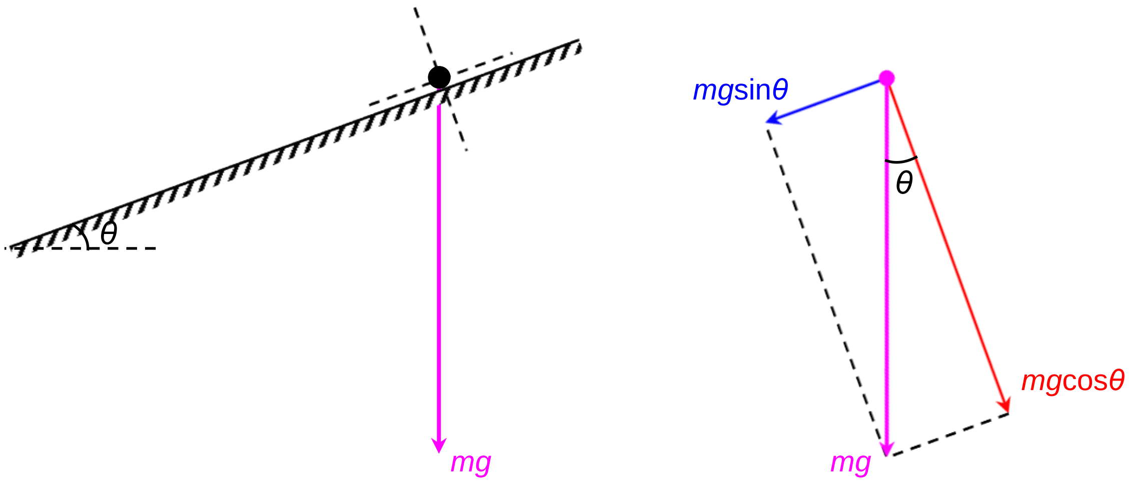
1.8.2 Resolving Vectors

In scalars, sometimes it is helpful to "break up" a number. For example, 27 is 20 + 7. In vectors we have a similar thing. We often "break" a vector into its two perpendicular components. We expect students to use the word "resolve", of course.

Resolving vectors is like the reverse of summation of two perpendicular vectors. As such, the vector which we want to resolve, forms the diagonal of the "rectangle". The adjacent and opposite sides of the "rectangle" are the two components we are looking for.



Above, the slanted tension force *T* is resolved into its vertical component *T*cos*θ* and horizontal component *T*sin*θ*.



Above, the vertical weight *mg* is resolved into its component parallel to the slope *mg*sin*θ* and component perpendicular to the slope *mg*cosθ.