XMLECTURE

01 MEASUREMENT

NO DEFINITIONS. JUST PHYSICS.

Online resources are provided at https://xmphysics.com/measurement/

1.1 SI Units

$1 \text{ mile} = 1760 \text{ yards} \times 3 \text{ feet} \times 12 \text{ inches} = 63{,}360 \text{ inches}$

$1 \text{ km} = 1000 \text{ m} \times 1000 \text{ mm} = 1{,}000{,}000 \text{ mm}$

Aren't you glad that most countries have adopted the metric system? But how does everyone come to agree on exactly how long a metre should be?

For that we turn to the International Bureau of Weight and Measures (BIPM), which administers the International System of Units (Le Système International d'Unités) or SI Units. In total, there are 7 standardized base units for 7 base quantities, as listed in the table below.

Base Quantity	Base Units	
Mass	Kilogram	kg
Length	Metre	m
Time	Second	s
Electric current	Ampere	A
Thermodynamic temperature	Kelvin	K
Amount of substance	Mole	mol
Luminous intensity	Candela	Cd

What about other units such as the Newton and the Joule? Well, if you think about it, N is actually kg m s^{-2} (think $F = ma$), and J is actually $\text{kg m}^2 \text{ s}^{-2}$ (think $KE = \frac{1}{2}mv^2$). Since they can be derived from base units, they are called the derived units of derived quantities. More examples are given in the table below.

Derived Quantity	Derived Unit	Think	Base Units
Force	N	$F = ma$	kg m s^{-2}
Energy	J	$KE = \frac{1}{2}mv^2$	$\text{kg m}^2 \text{ s}^{-2}$
Power	W	$E = Pt$	$\text{kg m}^2 \text{ s}^{-3}$
Charge	C	$I = \frac{Q}{t}$	A s
Voltage	V	$P = VI$	$\text{kg m}^2 \text{ s}^{-3} \text{ A}^{-1}$

1.2 Prefixes

Every day, I dream of winning the one-million-dollar lottery. Then I realize I have to win the lottery every year for 1000 years, before I match the wealth of a nincompoop like Donald Trump. That's what I call perspective.

Anyway, back to our lecture. Physicists poke their noses into everything, from the smallest to the largest. The diameter of a proton is 0.000 000 000 000 0016 m, while Proxima Centauri, the nearest star to Earth, is 40,208,000,000,000,000 m away. To avoid death from zeroes counting, we have invented the standard notation plus the system of prefixes.

Factor	Prefix	Symbol	Order of Magnitude
10^{-15}	femto	f	-15
10^{-12}	pico	p	-12
10^{-9}	nano	n	-9
10^{-6}	micro	u	-6
10^{-3}	milli	m	-3
10^{-2}	centi	c	-2
10^{-1}	deci	d	-1
10^{0}	-	-	0
10^{3}	kilo	k	3
10^{6}	mega	M	6
10^{9}	giga	G	9
10^{12}	tera	T	12

So, the diameter of a proton is 1.6 femtometer, while Proxima Centauri is 40 terameter away[1]. Much easier right?

powers of ten

[1] Actually, astronomers don't use Mm, Gm and Tm for astronomical distances. They prefer units like the astronomical unit (symbol AU, 1 AU = 149,597,871 km), based on the Sun-Earth-distance, and the lightyear (symbol ly, 1 ly = 9.46×10^{12} km), based on the distance travelled by light in one year.

1.3 Estimation

As physics students, you are required to make reasonable estimates. The expectation is that you should at least arrive at the correct order of magnitude.

For example, I have once asked my students to estimate the volume occupied by my body. Is it about 10 m^3, 1 m^3, 0.1 m^3 or 0.01 m^3? By the way, that's what we mean by orders of magnitude.

Student A, who must have watched some contortionist performance the previous night, says "I bet we can pack Mr. Chua into a $40\text{ cm}\times 40\text{ cm}\times 60\text{ cm}\approx 0.1\text{ m}^3$ box."
Student B, who must have gone swimming the previous day, says "Mr Chua can barely float in water. With a weight of around 100 kg, his volume should be about $V = M \div \rho = 100\text{ kg} \div 1\text{ g cm}^{-3} = 0.1\text{ m}^3$."
Student C, who must have just drunk 1-litre of coke during recess, says "$1\text{ m}^3 = 10^6\text{ cm}^3 = 1000\text{ litre}$. Mr Chua looks like 100 coke bottles to me, which is 100 litre, which is 0.1 m^3"

As you can see, estimation problems can be approached from different angles and can be a lot of fun. And having some ballpark figures in your head can be very helpful.

<u>Length</u>

Radius of Earth	6400 km
MRT train	100 m
Hair thickness	0.1 mm
Size of atom	10^{-10} m
Size of nucleus	10^{-15} m

<u>Speed</u>

Jogging	12 km h^{-1}
Speed limit on roads	50 km h^{-1}
Speed limit on expressways	80 km h^{-1}
Jet plane	900 km h^{-1}
Sound wave	330 m s^{-1}
Light wave	3.00×10^8 m s^{-1}

Volume

1 litre bottle	1000 cm^3
Can drink	330 ml

Mass

1 cm^3 of water	1 g
1 m^3 of water	1000 kg
Car	2000 kg
Earth	6.0×10^{24} kg

Density

Water	1 g cm^{-3}, or 1 ton per m^3
Ice	About 90% that of water's
Steel	About 8 times that of water's

Pressure

Atmospheric pressure	1.0×10^5 Pa.
Car tyre	200 kPa

Energy

Specific heat capacity of water	4.2 kJ kg^{-1}.
Specific latent heat of water	330 kJ kg^{-1}.
Specific latent heat of vaporization	2.3 MJ kg^{-1}.

Voltage

Duracell battery	1.5 V
Mains supply	220 V
CRT, X-ray machine	> 1 kV

Power

A domestic light bulb	10 W
Electric Kettle	~ 2 kW
Power plant	~ MW

1.4 Errors and Uncertainties

As a physics student, you must know the difference between errors and uncertainties.

Let's say, a student uses an electronic balance to weigh a 1 dollar coin. The reading shown on the balance is 7.6 g, which means that the actual mass could be anything between 7.55 and 7.65 g. With this in mind, he presents his measurement as 7.60 ± 0.05 g . The 0.05 g uncertainty is an **instrumental uncertainty** caused by the resolution of the measuring instrument.

Another student uses a stopwatch to measure the time taken for a ball to roll down a slope. The stopwatch shows a reading of 3.02 s. The student is aware that as a human there is inconsistency in the delays incurred during the starting and stopping of the stopwatch. So he wisely times the motion a few more times, obtaining readings ranging from 2.90 s to 3.10 s, with an averaged value of 3.00 s. With this in mind, he presents his measurement as 3.0 ± 0.1 s. The 0.1 s **procedural uncertainty** is caused by human reaction time.

When presenting of the measured value with its associated uncertainty, it is customary to

1. Round off the uncertainty to 1 significant figure only.
2. Round off the value to the same decimal place as the uncertainty.

1 s.f.

$$4.103 \pm 0.002$$

same d.p.

The examples below should clarify the above two points.

9.784 ± 0.028	is to be presented as 9.78 ± 0.03
1.234 ± 0.234	is to be presented as 1.2 ± 0.2
1234 ± 234	is to be presented as $1200 \pm 200 \quad \Rightarrow \quad (1.2 \pm 0.2) \times 10^{2}$
0.12351 ± 0.00098	is presented as $0.124 \pm 0.001 \quad \Rightarrow \quad (1.24 \pm 0.01) \times 10^{-1}$

So far, we have been talking only about uncertainties. To talk about errors, we have to know what the true or correct value is. Say for example, through a Google search, we learn that the "true" mass of the Singapore's dollar coin is 7.62 g. This means that the measurement of 7.60 g has an **error** of $7.60 - 7.62 = -0.02$ g. Or, through theoretical calculation, we obtain the "correct" value for the time needed for the ball to roll down that slope to be 2.8 s. The measurement of 3.0 s thus has an **error** of $3.0 - 2.8 = +0.2$ s.

1.5 Precision and Accuracy

"I would rather be imprecisely correct, than precisely wrong."

~ anonymous lazy experimenter

Roughly speaking, **precision** is related to the amount of uncertainty in a measurement. The smaller the uncertainty, the higher the precision. **Accuracy,** on the other hand, is related to the magnitude of error in a measurement. The smaller the error, the higher the accuracy.

Measurement A: $4.8 \pm 0.1\,\Omega$

Measurement B: $4.57 \pm 0.01\,\Omega$

True value: 4.70 Ω

In the above example, B is more precise than A (since it has an uncertainty of only 0.01 Ω compared to 0.1 Ω for A). But A is more accurate than B (since its error is only 0.1 Ω compared to −0.13 Ω for B)

These two concepts are also applicable when a measurement is repeated to obtain a data set.

Data set A = {1.000 s, **1.002** s, **0.998** s, 1.001 s, 0.999 s}   Average value, <A> = 1.000 s [2]

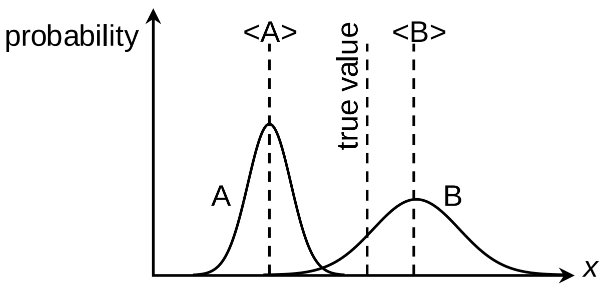
Data set B = {**1.032** s, 1.010 s, 1.001 s, 1.018 s, **0.989** s}   Average value, <B> = 1.010 s

True value: 1.008 s

For the example given above, A is more precise since its range ($1.002 - 0.998 = 0.004$ s) is narrower.

B however is more accurate since its error ($1.010 - 1.008 = 0.002$ s) is smaller.

If we have a large data set, we can even plot out the distribution curve. With a large population, the graph often shows a normal distribution. The precision will correspond to the width or spread of the graph (standard deviation would be the more technical term), while the accuracy is reflected in the deviation of the mean value from the true value.



In the example above, A is more precise, but B is more accurate.

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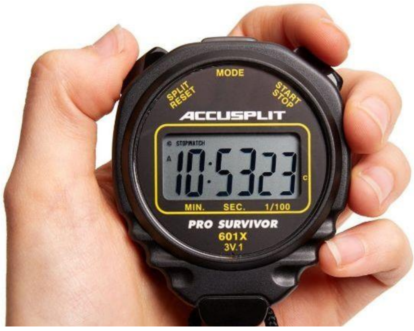
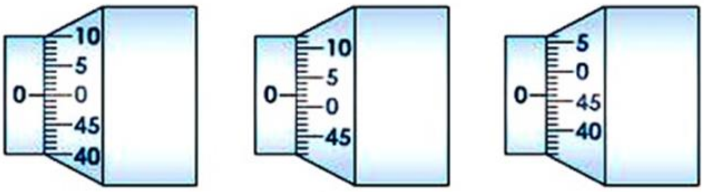
[2] <X> denotes average value of X

1.6 Random and Systematic Errors

Both random errors[3] and systematic errors lower the quality of our measurements. Basically, random uncertainties (whether instrumental or procedural) results in imprecision, while systematic errors causes inaccuracy. In this section, we will highlight the difference between these two sources of errors.



Random errors are random, unpredictable and therefore not reproducible. Timing the fall of a ball is a very good example of a measurement strewn with random errors. You make repeated measurements and get multiple different readings, and nobody knows for sure which reading is closest to the true value. But if random errors are equally likely to be positive and negative (in many practical situations, random errors are normally distributed), they should add up to zero statistically. For this reason, we are expected to repeat measurements and calculate the average. By repeat-and-averaging, we have a good statistical chance of reducing (we dare not say eliminate) the random errors and improve our chance of hitting the true value.

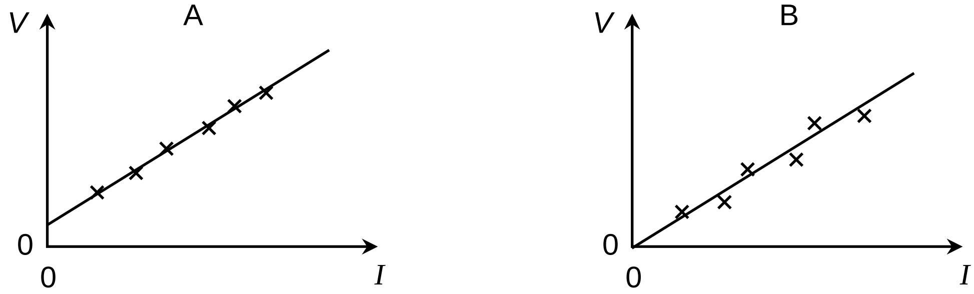


(a) No zero error (b) Positive zero error (c) Negative zero error

On the other hand, systematic errors are consistent, predictable and therefore reproducible. The zero error is a classic example. If your micrometer screw gauge has zero error of +0.02 mm, every single reading will be too large by 0.02 mm. Do you realize that systematic errors (unlike random errors) cannot be reduced by repeat-and-averaging? On the other hand, if we can identify the source of systematic errors, and take appropriate actions, we can eliminate them completely.

[3] Actually "random uncertainties" may be a more correct term. But as a community we are often guilty of using the word "error" when we actually mean "uncertainty".

Plotting our data on a graph allows us to literally see the random and systematic errors in our measurements. For example, when we plot the *V*-*I* graph of a resistor, we expect to obtain a straight line graph passing through the origin (because $V = IR$). We can also calculate the gradient of the graph to obtain the resistance of the resistor.



For the example shown above, the random errors are larger in B than A. This is indicated by the amount of scatter of the data points on both sides of the best-fit-line. But the line does not pass through the origin in A. This suggests the presence of systematic errors in A. (Perhaps the voltmeter has a positive zero error.)

1.7 Propagation of Error/Uncertainty

Let's say that we have measured the length and breadth of a rectangle to be $L = 2.00 \pm 0.01\text{ m}$ and $B = 1.00 \pm 0.01\text{ m}$. Everybody knows how to calculate the perimeter and area of the rectangle: $P = 6.00\text{ m}$ and $A = 2.00\text{ m}^2$. But surely *P* and *A* must "inherit" the uncertainties of *L* and *B*? So how do we calculate their "inheritances"?

Well, the brute force approach is to calculate the extreme limits, as shown below:

$$P_{max} = 2(2.01 + 1.01) = 6.04\text{ m}$$

$$P_{min} = 2(1.99 + 0.99) = 5.96\text{ m}$$

$$\text{So } P = (6.00 \pm 0.04)\text{ m}$$

$$A_{max} = (2.01 \times 1.01) = 2.0301\text{ m}^2$$

$$A_{min} = (1.99 \times 0.99) = 1.9701\text{ m}^2$$

$$\text{So } A = (2.00 \pm 0.03)\text{ m}^2$$

If you are sharp, you might have noticed that the (absolute) uncertainty of *P* is simply the summation of the (absolute) uncertainties of *L* and *B*.

$$P = 2L + 2B \;\Rightarrow\; \Delta P = 2\Delta L + 2\Delta B = 0.04\text{ m}$$

How about ΔA? Obviously $\Delta A \neq \Delta L + \Delta B$. The units don't even match. But if I point out to you that $\frac{\Delta A}{A} = \frac{0.03}{2.00} = 1.5\%$, $\frac{\Delta L}{L} = \frac{0.01}{2.00} = 0.5\%$ and $\frac{\Delta B}{B} = \frac{0.01}{1.00} = 1\%$, do you realize that we could have obtained the percentage uncertainty of *A* by adding up the percentage uncertainties of *L* and *B*?

$$A = L \times B \;\Rightarrow\; \frac{\Delta A}{A} = \frac{\Delta L}{L} + \frac{\Delta B}{B} = 1.5\% \;\Rightarrow\; \Delta A = 1.5\% \times A = 0.03\text{ m}^2$$

This trick works because if *L* increases slightly by *x*% and *B* slightly by y%, then *A* will increase by about $(x + y)\%$. If you know your binomial expansion, it's not difficult to understand the "coincidence" of $1.01 \times 1.02 \approx 1.03$, $1.02 \times 1.03 \approx 1.05$, etc.

Ok. That's the rough idea. Different approaches for different types of calculations. Let's now look at the details.

1. If $S = mA + nB + k, \quad \Delta S = |m|\Delta A + |n|\Delta B$

Basically, when we are summing up measurements, we can sum up the **absolute** uncertainties directly. The constant coefficients (*m* and *n*) provide the "scaling factors". The constant *k* has zero uncertainty because it is not a measured value.

The roles played by the constants can be illustrated with an example:

$$\begin{aligned} S &= 2A - 3B + 5, \\ S &= A + A - B - B - B + 5 \\ \Delta S &= \Delta A + \Delta A + \Delta B + \Delta B + \Delta B + 0 \\ &= 2\Delta A + 3\Delta B \end{aligned}$$

Note that the uncertainties always add up, even if the measurement is being subtracted. For example, if I earn $\$(3000 \pm 500)$ but spend $\$(1000 \pm 300)$ per month, my saving per month would range from $\$3500 - \$700 = \$2800$ down to $\$2500 - \$1300 = \$1200$. Or $\$(2000 \pm 800)$. See? The uncertainties always add up because we are going for the "worst" outcomes.

2. If $P = kA^m B^n, \quad \Rightarrow \quad \dfrac{\Delta P}{P} = |m|\dfrac{\Delta A}{A} + |n|\dfrac{\Delta B}{B}$

Basically, when we are multiplying measurements together, we sum up the **percentage** uncertainties first (then use it to calculate the absolute uncertainty). The constant powers (*m* and *n*) provide the "scaling factors". The constant coefficient *k* has zero percentage uncertainty because it is not a measured value.

The roles played by the constants can be illustrated with an example:

$$\begin{aligned} P &= 5A^2 \div B^3, \\ P &= 5 \times A \times A \div B \div B \div B \\ \frac{\Delta P}{P} &= 0 + \frac{\Delta A}{A} + \frac{\Delta A}{A} + \frac{\Delta B}{B} + \frac{\Delta B}{B} + \frac{\Delta B}{B} \\ &= 2\frac{\Delta A}{A} + 3\frac{\Delta B}{B} \end{aligned}$$

Note that the percentage uncertainties always add up, even if the measurement is being divided. For example, if the mass and volume of an object is measured to be $M = 1.00 \pm 1\%$ kg and $V = 1.00 \pm 2\%$ m^3 , then its density would range from $1.01 \div 0.98 = 1.03$ kg m^{-3} down to $0.99 \div 1.02 = 0.97$ kg m^{-3}. Or $1.00 \pm 3\%$ kg m^{-3}. See? The percentage uncertainties always add up because we are going after the extreme cases.

3. If calculation is complex $\Rightarrow$ $\Delta Z = \dfrac{Z_{max} - Z_{min}}{2}$

If the calculation is anything other than a pure summation or pure multiplication of terms (e.g. $Z = A^2 + B^2$), then applying the preceding two shortcuts can be very cumbersome. In such cases, it is much easier to just fall back to the "half-max-minus-min" method. This brute-force method should also be used when we encounter non-linear functions such as sin, cos, $\tan^{-1}$, log, etc.

E.g. if $H = L\cos\theta$, and L and θ are measured to be $L = 2.0 \pm 0.2\text{ cm}$ and $\theta = 30 \pm 2°$, then

$H_{max} = 2.2\cos 28° = 1.942$

$H_{min} = 1.8\cos 32° = 1.526$

$\Delta H = \dfrac{1.942 - 1.526}{2} = 0.208 \approx 0.2\text{ cm}$

1.8 Vectors

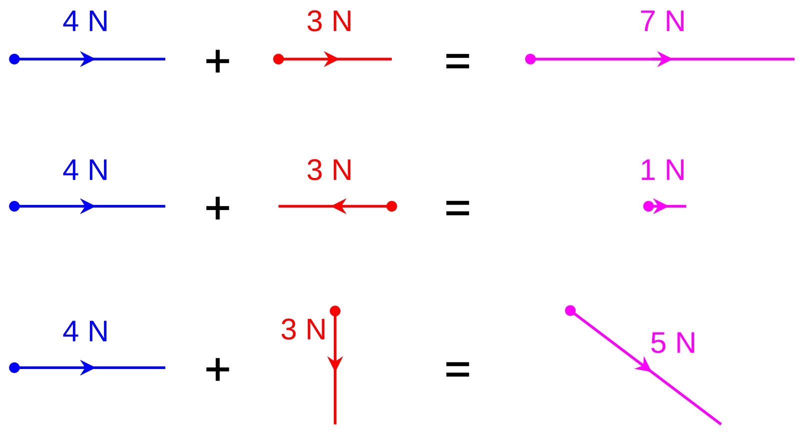
There are two types of quantities: scalar and vectors.

Scalars are those with magnitude only: distance, speed, mass, etc.
Vectors are those with both magnitude and direction: displacement, velocity, force, etc.

A common newbie mistake is to associate polarity with direction. For example, we have temperatures of +10°C and −10°C, and GPE can be +10 J or −10 J. But careful, temperature and energy are scalar quantities. There is no such thing as a rightward or leftward temperature. Neither is there a northward or southward GPE. Do you realize that the plus and minus signs here do not denote any spatial direction? Compare them with those used in proper vectors. For example, +1.0 m s^{-1} and −1.0 m s^{-1} may imply rightward and leftward velocities respectively. The plus and minus signs do indeed denote a spatial direction.

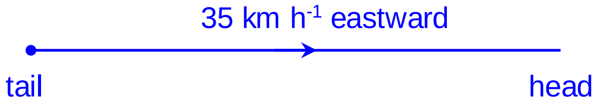
1.8.1 Vector Diagram

Scalar addition is straight forward: 3 + 4 is always equal to 7. Vector addition is slightly trickier. Take for example the summation of a 3 N and a 4 N force. If they are in the same direction, the resultant force is 7 N. If the two forces are in opposite directions, the resultant force is 1 N. If they are perpendicular to each other, the resultant is a 5 N. In fact, depending on the direction of the two vectors, 3 N + 4 N can be anything between 1 N and 7 N.

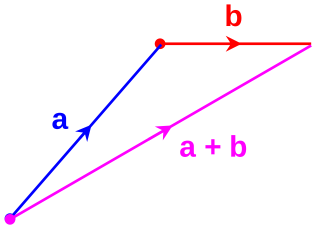


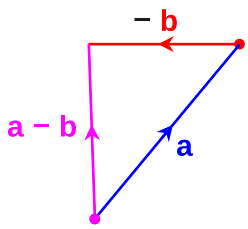
From your secondary school math, you may have learnt vector notations such as $\overrightarrow{AB}$, $\underset{\sim}{a} - \underset{\sim}{b}$ $\begin{pmatrix}3\\4\end{pmatrix}$, $3\mathbf{i} + 4\mathbf{j}$, etc. Well, you won't encounter any of these in A-level Physics. The most complicated thing you'll be made to do, is drawing a few arrows (called a vector diagram) and solving a few triangles.

Just to jog your memory. In vector diagrams, each vector is represented by an arrowed line: the length indicates the magnitude, the arrow indicates the direction.

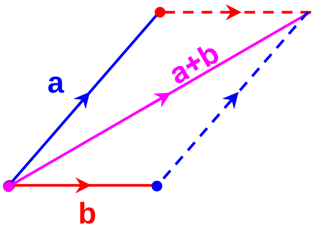


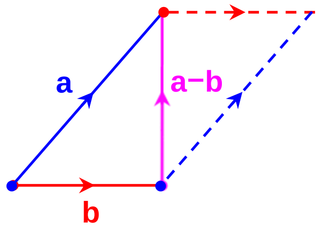
To add two vectors **a** and **b** together, we arrange **a** and **b** head-to-tail, and draw the resultant vector **a+b** from the free tail to the free head. To subtract **b** from **a**, we arrange **a** and **−b** (meaning we have to flip **b** 180° around) head-to-tail, and draw the resultant vector **a−b** from the free tail to the free head.



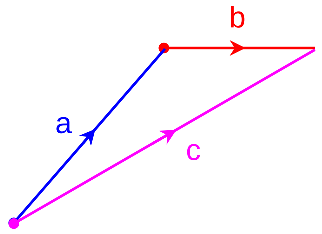


When you're more confident, you can also use the parallelogram method. By bringing the tails of **a** and **b** together (instead of head-to-tail), you can obtain **a** + **b** and **a** – **b** from the diagonals of the parallelogram.



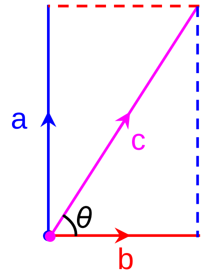


Technically, if you construct your vector diagrams accurately enough with rulers and protractors, you can measure the length of the resultant vector to obtain its magnitude (taking into account the scale of your diagram). But since you have learnt enough mathematics, it is more convenient to do a decent sketch of the vector diagrams but solve them accurately through calculations, typically using cosine rule or sine rule.



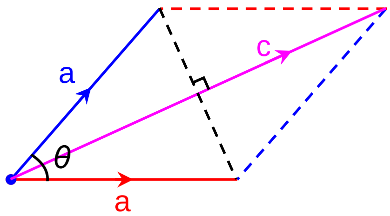
$$c^2 = a^2 + b^2 - 2ab\cos\hat{c}$$
$$\frac{a}{\sin\hat{a}} = \frac{b}{\sin\hat{b}} = \frac{c}{\sin\hat{c}}$$

Often, in H2 physics we are working with perpendicular vectors. So our parallelograms and triangles become rectangles and right-angled triangles. In these cases, we can dump the sine and cosine rules, and use Pythagoras and trigo ratios instead.



$$c^2 = a^2 + b^2$$
$$\tan\theta = \frac{a}{b}$$

At other times, we are working with vectors of the same magnitude, in which case our parallelograms and triangles become rhombuses and isosceles triangles, which are again relatively easy to solve.

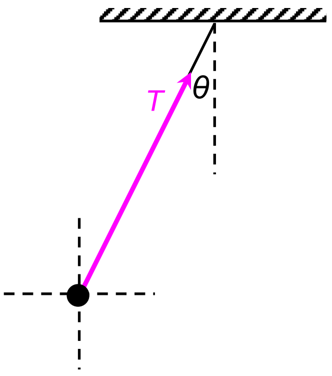


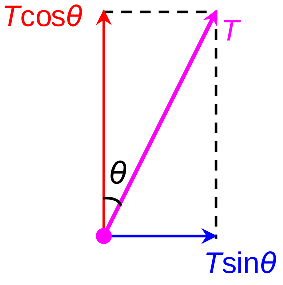
$$c = 2a\cos\frac{\theta}{2}$$

1.8.2 Resolving Vectors

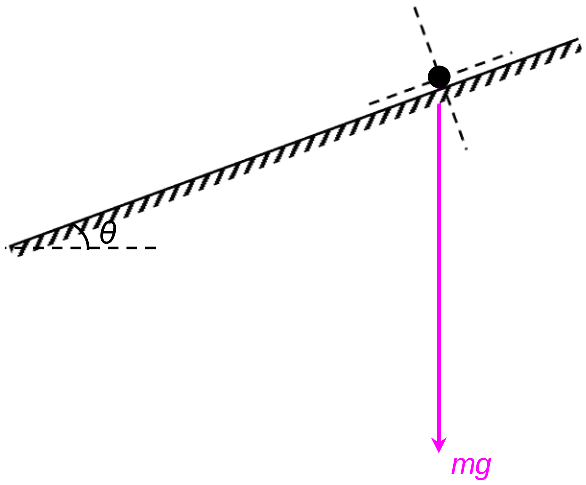
In scalars, sometimes it is helpful to "break up" a number. For example, 27 is 20 + 7. In vectors we have a similar thing. We often "break" a vector into its two perpendicular components. This vector is then said to be resolved.

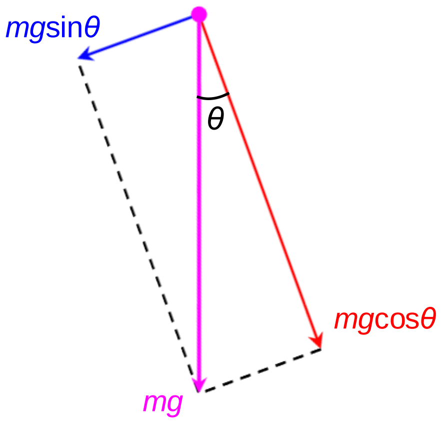
Resolving vectors is actually the reverse of the summation of two perpendicular vectors. As such, the vector which we want to resolve, must form the diagonal of the "rectangle". The adjacent and opposite sides of the "rectangle" are the two components we are looking for.





As shown above, the slanted tension force T is resolved into its vertical component $T\cos\theta$ and horizontal component $T\sin\theta$.





As shown above, the vertical weight mg is resolved into its component parallel to the slope $mg\sin\theta$ and component perpendicular to the slope $mg\cos\theta$.